Chapter 3

Dynamic Response

Problems and Solutions for Section 3.1

1. Show that, in a partial-fraction expansion, complex conjugate poles have coefficients that are also complex conjugates. (The result of this relationship is that whenever complex conjugate pairs of poles are present, only one of the coefficients needs to be computed.)

   Solution:

   Consider the second-order system with poles at $-\alpha \pm j\beta$,

$$
\begin{aligned}
H(s) &= \frac{1}{(s+\alpha+j\beta)(s+\alpha-j\beta)} \\
&\quad \text{P erf orm Partial Fraction Expansion:} \\
H(s) &= \frac{C_1}{s+\alpha+j\beta} + \frac{C_2}{s+\alpha-j\beta} \\
C_1 &= \frac{1}{s+\alpha-j\beta}|_{s=-\alpha-j\beta} = \frac{1}{2\beta}j \\
C_2 &= \frac{1}{s+\alpha+j\beta}|_{s=-\alpha+j\beta} = -\frac{1}{2\beta}j \\
\therefore &\quad C_1 = C_2^*
\end{aligned}
$$

2. Find the Laplace transform of the following time functions:

   (a) $f(t) = 1 + 2t$

   (b) $f(t) = 3 + 7t + t^2 + \delta(t)$

   (c) $f(t) = e^{-t} + 2e^{-2t} + te^{-3t}$

   (d) $f(t) = (t+1)^2$

   (e) $f(t) = \sinh t$

Solution:

(a)

$$\begin{aligned} f(t) &= 1+2t \\ \mathcal{L}\{f(t)\} &= \mathcal{L}\{1(t)\}+\mathcal{L}\{2t\} \\ &= \frac{1}{s}+\frac{2}{s^2} \\ &= \frac{s+2}{s^2} \end{aligned}$$

(b)

$$\begin{aligned} f(t) &= 3+7t+t^2+\delta(t) \\ \mathcal{L}\{f(t)\} &= \mathcal{L}\{3\}+\mathcal{L}\{7t\}+\mathcal{L}\{t^2\}+\mathcal{L}\{\delta(t)\} \\ &= \frac{3}{s}+\frac{7}{s^2}+\frac{2!}{s^3}+1 \\ &= \frac{s^3+3s^2+7s+2}{s^3} \end{aligned}$$

(c)

$$\begin{aligned} f(t) &= e^{-t}+2e^{-2t}+te^{-3t} \\ \mathcal{L}\{f(t)\} &= \mathcal{L}\{e^{-t}\}+\mathcal{L}\{2e^{-2t}\}+\mathcal{L}\{te^{-3t}\} \\ &= \frac{1}{s+1}+\frac{2}{s+2}+\frac{1}{(s+3)^2} \end{aligned}$$

(d)

$$\begin{aligned} f(t) &= (t+1)^2 \\ &= t^2+2t+1 \\ \mathcal{L}\{f(t)\} &= \mathcal{L}\{t^2\}+\mathcal{L}\{2t\}+\mathcal{L}\{1\} \\ &= \frac{2!}{s^3}+\frac{2}{s^2}+\frac{1}{s} \\ &= \frac{s^2+2s+2}{s^3} \end{aligned}$$

(e) Using the trigonometric identity,

$$\begin{aligned} f(t) &= \sinh t \\ &= \frac{e^t-e^{-t}}{2} \\ \mathcal{L}\{f(t)\} &= \mathcal{L}\{\frac{e^t}{2}\}-\mathcal{L}\{\frac{e^{-t}}{2}\} \\ &= \frac{1}{2}(\frac{1}{s-1})-\frac{1}{2}(\frac{1}{s+1}) \\ &= \frac{1}{s^2-1} \end{aligned}$$

3. Find the Laplace transform of the following time functions:

   (a) $f(t) = 3\cos 6t$

   (b) $f(t) = \sin 2t + 2\cos 2t + e^{-t}\sin 2t$

   (c) $f(t) = t^2 + e^{-2t}\sin 3t$

   Solution:

   (a)

$$\begin{aligned} f(t) &= 3\cos 6t \\ \mathcal{L}\{f(t)\} &= \mathcal{L}\{3\cos 6t\} \\ &= 3\frac{s}{s^2+36} \end{aligned}$$

   (b)

$$\begin{aligned} f(t) &= \sin 2t + 2\cos 2t + e^{-t}\sin 2t \\ &= \mathcal{L}\{f(t)\} = \mathcal{L}\{\sin 2t\} + \mathcal{L}\{2\cos 2t\} + \mathcal{L}\{e^{-t}\sin 2t\} \\ &= \frac{2}{s^2+4} + \frac{2s}{s^2+4} + \frac{2}{(s+1)^2+4} \end{aligned}$$

   (c)

$$\begin{aligned} f(t) &= t^2 + e^{-2t}\sin 3t \\ &= \mathcal{L}\{f(t)\} = \mathcal{L}\{t^2\} + \mathcal{L}\{e^{-2t}\sin 3t\} \\ &= \frac{2!}{s^3} + \frac{3}{(s+2)^2+9} \\ &= \frac{2}{s^3} + \frac{3}{(s+2)^2+9} \end{aligned}$$

4. Find the Laplace transform of the following time functions:

   (a) $f(t) = t\sin t$

   (b) $f(t) = t\cos 3t$

   (c) $f(t) = te^{-t} + 2t\cos t$

   (d) $f(t) = t\sin 3t - 2t\cos t$

   (e) $f(t) = 1(t) + 2t\cos 2t$

   Solution:

   (a)

$$\begin{aligned} f(t) &= t\sin t \\ \mathcal{L}\{f(t)\} &= \mathcal{L}\{t\sin t\} \end{aligned}$$

Use multiplication by time Laplace transform property (Table A.1, entry #11),

$$
\begin{aligned}
\mathcal{L}\{tg(t)\} &= -\frac{d}{ds}G(s) \\
\text{Let } g(t) &= \sin t \text{ and use } \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2} \\
\mathcal{L}\{t\sin t\} &= -\frac{d}{ds}(\frac{1}{s^2+1^2}) \\
&= \frac{2s}{(s^2+1)^2} \\
&= \frac{2s}{s^4+2s^2+1}
\end{aligned}
$$

(b)

$$f(t) = t\cos 3t$$

Use multiplication by time Laplace transform property (Table A.1, entry #11),

$$
\begin{aligned}
\mathcal{L}\{tg(t)\} &= -\frac{d}{ds}G(s) \\
\text{Let } g(t) &= \cos 3t \text{ and use } \mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2} \\
\mathcal{L}\{t\cos 3t\} &= -\frac{d}{ds}(\frac{s}{s^2+9}) \\
&= \frac{-[(s^2+9)-(2s)s]}{(s^2+9)^2} \\
&= \frac{s^2-9}{s^4+18s^2+81}
\end{aligned}
$$

(c)

$$f(t) = te^{-t} + 2t\cos t$$

Use the following Laplace transforms and properties (Table A.1, entries 4,11, and 3),

$$\begin{aligned}
\mathcal{L}\{te^{-at}\} &= \frac{1}{(s+a)^2} \\
\mathcal{L}\{tg(t)\} &= -\frac{d}{ds}G(s) \\
\mathcal{L}\{\cos at\} &= \frac{s}{s^2+a^2} \\
\mathcal{L}\{f(t)\} &= \mathcal{L}\{te^{-t}\} + 2\mathcal{L}\{t\cos t\} \\
&= \frac{1}{(s+1)^2} + 2(-\frac{d}{ds}\frac{s}{s^2+1}) \\
&= \frac{1}{(s+1)^2} - 2\left[\frac{(s^2+1)-(2s)s}{(s^2+1)^2}\right] \\
&= \frac{2s^2-1}{s^4+2s^2+1}
\end{aligned}$$

(d)

$$f(t) = t\sin 3t - 2t\cos t$$

Use the following Laplace transforms and properties (Table A.1, entries 11, 3),

$$\begin{aligned}
\mathcal{L}\{tg(t)\} &= -\frac{d}{ds}G(s) \\
\mathcal{L}\{\sin at\} &= \frac{a}{s^2+a^2} \\
\mathcal{L}\{\cos at\} &= \frac{s}{s^2+a^2} \\
\mathcal{L}\{f(t)\} &= \mathcal{L}\{t\sin 3t\} - 2\mathcal{L}\{t\cos t\} \\
&= -\frac{d}{ds}\frac{3}{s^2+9} - 2(-\frac{d}{ds}\frac{s}{s^2+1}) \\
&= \frac{-(2s*3)}{(s^2+9)^2} - 2\frac{((s^2+1)-(2s)s)}{(s^2+1)^2} \\
&= \frac{-6s}{(s^2+9)^2} + \frac{2(s^2-1)}{(s^2+1)^2}
\end{aligned}$$

(e)

$$\begin{aligned}
f(t) &= 1(t) + 2t\cos 2t \\
\mathcal{L}\{1(t)\} &= \frac{1}{s} \\
\mathcal{L}\{tg(t)\} &= -\frac{d}{ds}G(s) \\
\mathcal{L}\{\cos at\} &= \frac{s}{s^2+a^2} \\
\mathcal{L}\{f(t)\} &= \mathcal{L}\{1(t)\} + 2\mathcal{L}\{t\cos 2t\} \\
&= \frac{1}{s} + 2(-\frac{d}{ds}\frac{s}{s^2+4}) \\
&= \frac{1}{s} - 2\left[\frac{(s^2+4)-(2s)s}{(s^2+4)^2}\right] \\
&= \frac{1}{s} - 2\frac{(-s^2+4)}{(s^2+4)^2}
\end{aligned}$$

5. Find the Laplace transform of the following time functions (* denotes convolution):

   (a) $f(t) = \sin t \sin 3t$
   (b) $f(t) = \sin^2 t + 3\cos^2 t$
   (c) $f(t) = (\sin t)/t$
   (d) $f(t) = \sin t * \sin t$
   (e) $f(t) = \int_0^t \cos(t-\tau)\sin\tau d\tau$

   Solution:

   (a)

$$f(t) = \sin t \sin 3t$$

Use the trigonometric relation,

$$\begin{aligned}
\sin\alpha t\sin\beta t &= \frac{1}{2}\cos(|\alpha-\beta|t) - \frac{1}{2}\cos(|\alpha+\beta|t) \\
\alpha &= 1 \text{ and } \beta = 3 \\
f(t) &= \frac{1}{2}\cos(|1-3|t) - \frac{1}{2}\cos(|1+3|t) \\
&= \frac{1}{2}\cos 2t - \frac{1}{2}\sin 4t \\
\mathcal{L}\{f(t)\} &= \frac{1}{2}\mathcal{L}\{\cos 2t\} - \frac{1}{2}\mathcal{L}\{\cos 4t\} \\
&= \frac{1}{2}[\frac{s}{s^2+4} - \frac{s}{s^2+16}] \\
&= \frac{6s}{(s^2+4)(s^2+16)}
\end{aligned}$$

(b)

$$f(t) = \sin^2 t + 3\cos^2 t$$

Use the trigonometric formulas,

$$\begin{aligned}
\sin^2 t &= \frac{1-\cos 2t}{2} \\
\cos^2 t &= \frac{1+\cos 2t}{2} \\
f(t) &= \frac{1-\cos 2t}{2} + 3(\frac{1+\cos 2t}{2}) \\
&= 2 + \cos 2t \\
\mathcal{L}\{f(t)\} &= \mathcal{L}\{2\} + \mathcal{L}\{\cos 2t\} \\
&= \frac{2}{s} + \frac{s}{s^2+4} \\
&= \frac{3s^2+8}{s(s^2+4)}
\end{aligned}$$

(c) We first show the result that division by time is equivalent to integration in the frequency domain. This can be done as follows,

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t)dt \\
\int_s^\infty F(s)ds &= \int_s^\infty \left[\int_0^\infty e^{-st} f(t)dt\right] ds \\
&\text{Interchanging the order of integration,} \\
\int_s^\infty F(s)ds &= \int_0^\infty \left[\int_s^\infty e^{-st} ds\right] f(t)dt \\
\int_s^\infty F(s)ds &= \int_0^\infty \left[-\frac{1}{t}e^{-st}\right]_s^\infty f(t)dt \\
&= \int_0^\infty \frac{f(t)}{t} e^{-st} dt \\
&\text{U sin g this result then,}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}\{\sin t\} &= \frac{1}{s^2+1}, \\
\mathcal{L}\{\frac{\sin t}{t}\} &= \int_s^\infty \frac{1}{\xi^2+1} d\xi \\
&= \tan^{-1}(\infty) - \tan^{-1}(s) \\
&= \frac{\pi}{2} - \tan^{-1}(s) \\
&= \tan^{-1}(\frac{1}{s})
\end{aligned}$$

where a table of integrals was used and the last simplification follows from the related trigonometric identity.

(d)

$$f(t) = \sin t * \sin t$$

Use the convolution Laplace transform property (Table A.1, entry 7),

$$\begin{aligned}\mathcal{L}\{\sin t * \sin t\} &= (\frac{1}{s^2+1})(\frac{1}{s^2+1}) \\ &= \frac{1}{s^4+2s^2+1}\end{aligned}$$

(e)

$$\begin{aligned}f(t) &= \int_0^t \cos(t-\tau)\sin\tau d\tau \\ \mathcal{L}\{f(t)\} &= \mathcal{L}\{\int_0^t \cos(t-\tau)\sin\tau d\tau\} = \mathcal{L}\{\cos(t) * \sin(t)\}\end{aligned}$$

This is just the definition of the convolution theorem,

$$\begin{aligned}\mathcal{L}\{f(t)\} &= \frac{s}{s^2+1}\frac{1}{s^2+1} \\ &= \frac{s}{s^4+2s^2+1}\end{aligned}$$

6. Given that the Laplace transform of f(t) is F(s), find the Laplace transform of the following:

   (a) $g(t) = f(t)\cos t$

   (b) $g(t) = \int_0^t \int_0^{t_1} f(\tau) d\tau dt_1$

   Solution:

   (a) First write $\cos t$ in terms of the related Euler identity (Eq. B.33),

   $$g(t) = f(t)\cos t = f(t)\frac{e^{jt}+e^{-jt}}{2} = \frac{1}{2}f(t)e^{jt} + \frac{1}{2}f(t)e^{-jt}$$

   Then using entry 4 of Table A.1 we have,

   $$G(s) = \frac{1}{2}F(s-j) + \frac{1}{2}F(s+j) = \frac{1}{2}[F(s-j)+F(s+j)].$$

   (b) Let us define $\widetilde{f}(t_1) = \int_0^{t_1} f(\tau)d\tau$, then $g(t) = \int_0^t \widetilde{f}(t_1)dt_1$, and from entry 6 of Table A.1 we have $L\{\widetilde{f}(t)\} = \widetilde{F}(s) = \frac{1}{s}F(s)$ and using the same result again, we have $G(s) = \frac{1}{s}\widetilde{F}(s) = \frac{1}{s}(\frac{1}{s}F(s)) = \frac{1}{s^2}F(s)$

7. Find the time function corresponding to each of the following Laplace transforms using partial fraction expansions:

(a) $F(s) = \frac{2}{s(s+2)}$

(b) $F(s) = \frac{10}{s(s+1)(s+10)}$

(c) $F(s) = \frac{3s+2}{s^2+4s+20}$

(d) $F(s) = \frac{3s^2+9s+12}{(s+2)(s^2+5s+11)}$

(e) $F(s) = \frac{1}{s^2+4}$

(f) $F(s) = \frac{2(s+2)}{(s+1)(s^2+4)}$

(g) $F(s) = \frac{s+1}{s^2}$

(h) $F(s) = \frac{1}{s^6}$

(i) $F(s) = \frac{4}{s^4+4}$

(j) $F(s) = \frac{e^{-s}}{s^2}$

Solution:

(a) Perform partial fraction expansion,

$$
\begin{aligned}
F(s) &= \frac{2}{s(s+2)} \\
&= \frac{C_1}{s} + \frac{C_2}{s+2} \\
C_1 &= \frac{2}{s+2}|_{s=0} = 1 \\
C_2 &= \frac{2}{s}|_{s=-2} = -1 \\
F(s) &= \frac{1}{s} - \frac{1}{s+2} \\
\mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\{\frac{1}{s}\} - \mathcal{L}^{-1}\{\frac{1}{s+2}\} \\
f(t) &= 1(t) - e^{-2t}1(t)
\end{aligned}
$$

(b) Perform partial fraction expansion,

$$\begin{aligned}
F(s) &= \frac{10}{s(s+1)(s+10)} \\
&= \frac{C_1}{s} + \frac{C_2}{s+1} + \frac{C_3}{s+10} \\
C_1 &= \frac{10}{(s+1)(s+10)}|_{s=0} = 1 \\
C_2 &= \frac{10}{s(s+10)}|_{s=-1} = -\frac{10}{9} \\
C_3 &= \frac{10}{s(s+1)}|_{s=-10} = \frac{1}{9} \\
F(s) &= \frac{1}{s} - \frac{\frac{10}{9}}{s+1} + \frac{\frac{1}{9}}{s+10} \\
f(t) &= \mathcal{L}^{-1}\{F(s)\} = 1(t) - \frac{10}{9}e^{-t}1(t) + \frac{1}{9}e^{-10t}1(t)
\end{aligned}$$

(c) Re-write and carry out partial fraction expansion,

$$\begin{aligned}
F(s) &= \frac{3s+2}{s^2+4s+20} \\
&= 3\frac{(s+2)-\frac{4}{3}}{(s+2)^2+4^2} \\
&= \frac{3(s+2)}{(s+2)^2+4^2} - \frac{4}{(s+2)^2+4^2} \\
f(t) &= \mathcal{L}^{-1}\{F(s)\} = (3e^{-2t}\cos 4t - 4e^{-2t}\sin 4t)1(t)
\end{aligned}$$

(d) Perform partial fraction expansion,

$$\begin{aligned}
F(s) &= \frac{3s^2+9s+12}{(s+2)(s^2+5s+11)} \\
&= \frac{C_1}{s+2} + \frac{C_2 s + C_3}{s^2+5s+11} \\
C_1 &= \frac{3(s^2+3s+4)}{(s^2+5s+11)}|_{s=-2} = \frac{6}{5}
\end{aligned}$$

Equate numerators:

$$\frac{\frac{6}{5}}{(s+2)} + \frac{C_2 s + C_3}{(s^2+5s+11)} = \frac{3s^2+9s+12}{(s+2)(s^2+5s+11)}$$

$$(C_2 + \frac{6}{5})s^2 + (6 + C_3 + 2C_2)s + (2C_3 + \frac{66}{5}) = 3s^2 + 9s + 12$$

Equate like powers of s to find C_2 and C_3:

$$\begin{aligned}
C_2 + \frac{6}{5} &= 3 \Rightarrow C_2 = \frac{9}{5} \\
2C_3 + \frac{66}{5} &= 12 \Rightarrow C_3 = -\frac{3}{5} \\
F(s) &= \frac{\frac{6}{5}}{(s+2)} + \frac{\frac{9}{5}s - \frac{3}{5}}{(s^2+5s+11)} \\
&= \frac{\frac{6}{5}}{(s+2)} + \frac{s+\frac{5}{2}}{(s+\frac{5}{2})^2 + \frac{19}{4}} - \frac{17\sqrt{19}}{57}\frac{\frac{\sqrt{19}}{2}}{(s+\frac{5}{2})^2 + (\frac{\sqrt{19}}{2})^2} \\
f(t) &= \mathcal{L}^{-1}\{F(s)\} = (\frac{6}{5}e^{-2t} + e^{-\frac{5}{2}t}\cos\frac{\sqrt{19}}{2} + e^{-\frac{5}{2}t}\sin\frac{\sqrt{19}}{2})1(t)
\end{aligned}$$

(e) Re-write and use entry #17 of Table A.2,

$$\begin{aligned}
F(s) &= \frac{1}{s^2+4} \\
&= \frac{1}{2}\frac{2}{(s^2+2^2)} \\
f(t) &= \frac{1}{2}\sin 2t
\end{aligned}$$

(f)

$$\begin{aligned}
F(s) &= \frac{2(s+2)}{(s+1)(s^2+4)} \\
&= \frac{C_1}{(s+1)} + \frac{C_2 s + C_3}{(s^2+4)} \\
C_1 &= \frac{2(s+2)}{(s^2+4)}|_{s=-1} = \frac{2}{5}
\end{aligned}$$

Equate numerators and like powers of s terms:

$$\begin{aligned}
(\frac{2}{5} + C_2)s^2 + (C_2 + C_3)s + (\frac{8}{5} + C_3) &= 2s + 4 \\
\frac{8}{5} + C_3 &= 4 \qquad \Rightarrow C_3 = \frac{12}{5} \\
C_2 + C_3 &= 2 \qquad \Rightarrow C_2 = -\frac{2}{5} \\
\frac{2}{5} + C_2 &= 0
\end{aligned}$$

$$\begin{aligned}
F(s) &= \frac{\frac{2}{5}}{(s+1)} + \frac{-\frac{2}{5}s + \frac{12}{5}}{(s^2+4)} \\
&= \frac{\frac{2}{5}}{(s+1)} + \frac{-\frac{2}{5}s}{(s^2+2^2)} + \frac{6}{5}\frac{2}{(s^2+2^2)} \\
f(t) &= \frac{2}{5}e^{-t} - \frac{2}{5}\cos 2t + \frac{6}{5}\sin 2t
\end{aligned}$$

(g) Perform partial fraction expansion,

$$\begin{aligned}
F(s) &= \frac{s+1}{s^2} \\
&= \frac{1}{s} + \frac{1}{s^2} \\
f(t) &= (1+t)1(t)
\end{aligned}$$

(h) Use entry #6 of Table A.2,

$$\begin{aligned}
F(s) &= \frac{1}{s^6} \\
f(t) &= \mathcal{L}^{-1}\{\frac{1}{s^6}\} = \frac{t^5}{5!} = \frac{t^5}{60}
\end{aligned}$$

(i) Re-write as,

$$\begin{aligned}
F(s) &= \frac{4}{s^4+4} \\
&= \frac{\frac{1}{2}s+1}{s^2+2s+2} + \frac{-\frac{1}{2}s+1}{s^2-2s+2} \\
&= \frac{(s+1)-\frac{1}{2}s}{(s+1)^2+1} - \frac{(s-1)-\frac{1}{2}s}{(s-1)^2+1}
\end{aligned}$$

Use Table A.2 entry #19 and Table A.1 entry #5,

$$\begin{aligned}
f(t) &= \mathcal{L}^{-1}\{F(s)\} = e^{-t}\cos(t) - \frac{1}{2}\frac{d}{dt}\{e^{-t}\sin(t)\} - e^{t}\cos(t) \\
&\quad -\frac{1}{2}\frac{d}{dt}\{e^{t}\sin(t)\} \\
&= e^{-t}\cos(t) - \frac{1}{2}\{-e^{-t}\sin(t) + \cos(t)e^{-t}\} \\
&\quad -e^{t}\cos(t) + \frac{1}{2}\{e^{t}\sin(t) + \cos(t)e^{t}\} \\
&= -\cos(t)\{\frac{-e^{-t}+e^{t}}{2}\} + \sin(t)\{\frac{-e^{-t}+e^{t}}{2}\} \\
f(t) &= -\cos(t)\sinh(t) + \sin(t)\cosh(t)
\end{aligned}$$

(j) Using entry #2 of Table A.1,

$$\begin{aligned} F(s) &= \frac{e^{-s}}{s^2} \\ f(t) &= \mathcal{L}^{-1}\{F(s)\} = (t-1)1(t) \end{aligned}$$

8. Find the time function corresponding to each of the following Laplace transforms:

(a) $F(s) = \frac{1}{s(s+2)^2}$

(b) $F(s) = \frac{2s^2+s+1}{s^3-1}$

(c) $F(s) = \frac{2(s^2+s+1)}{s(s+1)^2}$

(d) $F(s) = \frac{s^3+2s+4}{s^4-16}$

(e) $F(s) = \frac{2(s+2)(s+5)^2}{(s+1)(s^2+4)^2}$

(f) $F(s) = \frac{(s^2-1)}{(s^2+1)^2}$

(g) $F(s) = \tan^{-1}(\frac{1}{s})$

Solution:

(a) Perform partial fraction expansion,

$$\begin{aligned} F(s) &= \frac{1}{s(s+2)^2} \\ &= \frac{C_1}{s} + \frac{C_2}{(s+2)} + \frac{C_3}{(s+2)^2} \\ C_1 &= sF(s)|_{s=0} = \frac{1}{(s+2)^2}|_{s=0} = \frac{1}{4} \\ C_3 &= (s+2)^2F(s)|_{s=-2} = \frac{1}{s}|_{s=-2} = -\frac{1}{2} \\ C_2 &= \frac{d}{ds}[(s+2)^2F(s)]_{s=-2} \\ &= \frac{d}{ds}[s^{-1}]_{s=-2} \\ &= -\frac{1}{s^2}|_{s=-2} \\ &= -\frac{1}{4} \\ F(s) &= \frac{\frac{1}{4}}{s} + \frac{-\frac{1}{4}}{(s+2)} + \frac{-\frac{1}{2}}{(s+2)^2} \\ f(t) &= \mathcal{L}^{-1}\{F(s)\} = (\frac{1}{4} - \frac{1}{4}e^{-2t} - \frac{1}{2}te^{-2t})1(t) \end{aligned}$$

(b) Perform partial fraction expansion,

$$
\begin{aligned}
F(s) &= \frac{2s^2+s+1}{s^3-1} \\
&= \frac{2s^2+s+1}{(s-1)(s^2+s+1)} \\
&= \frac{C_1}{s-1} + \frac{C_2 s + C_3}{s^2+s+1} \\
C_1 &= (s-1)F(s)|_{s=1} = \frac{2s^2+s+1}{s^2+s+1}|_{s=1} = \frac{4}{3}
\end{aligned}
$$

Equate numerators and like powers of s:

$$
\begin{aligned}
\frac{\frac{4}{3}}{s-1} + \frac{C_2 s + C_3}{s^2+s+1} &= \frac{2s^2+s+1}{(s-1)(s^2+s+1)} \\
s^2(\frac{4}{3}+C_2) + s(\frac{4}{3}-C_2+C_3) + (\frac{4}{3}-C_3) &= 2s^2+s+1 \\
\frac{4}{3}+C_2 &= 2 \qquad \Rightarrow C_2 = \frac{2}{3} \\
\frac{4}{3}-C_3 &= 1 \qquad \Rightarrow C_3 = \frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
F(s) &= \frac{\frac{4}{3}}{s-1} + \frac{\frac{2}{3}s+\frac{1}{3}}{s^2+s+1} \\
&= \frac{\frac{4}{3}}{s-1} + \frac{2}{3}\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} \\
f(t) &= \mathcal{L}^{-1}\{F(s)\} = \frac{4}{3}e^t + \frac{2}{3}e^{-\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t \\
&= \frac{2}{3}\{2e^t + e^{-\frac{t}{2}}\cos\frac{\sqrt{3}}{2}t\}1(t)
\end{aligned}
$$

(c) Carry out partial fraction expansion,

$$
\begin{aligned}
F(s) &= \frac{2(s^2+s+1)}{s\,(s+1)^2} \\
&= \frac{C_1}{s} + \frac{C_2}{(s+1)} + \frac{C_3}{(s+1)^2} \\
C_1 &= sF(s)|_{s=0} = \frac{2(s^2+s+1)}{(s+1)^2}|_{s=0} = 2 \\
C_3 &= (s+1)^2 F(s)|_{s=-1} = \frac{2(s^2+s+1)}{s}|_{s=-1} = -2 \\
C_2 &= \frac{d}{ds}[(s+1)^2 F(s)]_{s=-1} \\
&= \frac{d}{ds}[\frac{2(s^2+s+1)}{s}]_{s=-1} \\
&= \frac{2(2s+1)s - 2(s^2+s+1)}{s^2}|_{s=-1} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
F(s) &= \frac{2}{s} + \frac{0}{(s+1)} + \frac{-2}{(s+1)^2} \\
f(t) &= \mathcal{L}^{-1}\{F(s)\} = 2\{1 - te^{-t}\}1(t)
\end{aligned}
$$

(d) Carry out partial fraction expansion,

$$
\begin{aligned}
F(s) &= \frac{s^3+2s+4}{s^4-16} = \frac{As+B}{s^2-4} + \frac{Cs+D}{s^2+4} = \frac{\frac{3}{4}s+\frac{1}{2}}{s^2-4} + \frac{\frac{1}{4}s-\frac{1}{2}}{s^2+4} \\
&= \frac{1}{4}\sinh(2t) + \frac{3}{4}\frac{d}{dt}\{\frac{1}{2}\sinh(2t)\} - \frac{1}{4}\sin(2t) - \frac{1}{4}\frac{d}{dt}\{\frac{1}{2}\sin(2t)\} \\
&= \frac{1}{4}\sinh(2t) + \frac{3}{4}\cosh(2t) - \frac{1}{4}\sin(2t) + \frac{1}{4}\cos(2t)
\end{aligned}
$$

(e) Expand in partial fraction expansion and compute the residues using the results from Appendix A (pages 822-823),

$$\begin{aligned}
F(s) &= \frac{2(s+2)(s+5)^2}{(s+1)(s^2+4)^2} \\
&= \frac{C_1}{s+1} + \frac{C_2}{s-2j} + \frac{C_3}{s+2j} + \frac{C_4}{(s-2j)^2} + \frac{C_5}{(s+2j)^2} \\
C_1 &= (s+1)F(s)|_{s=-1} = \frac{32}{25} = 1.280 \\
C_4 &= (s-2j)^2F(s)|_{s=2j} = \frac{-83-39j}{20} = -4.150 - j1.950 \\
C_5 &= C_4^* = -4.150 + j1.950 \\
C_2 &= \frac{d}{ds}\left[(s-2j)^2F(s)\right]|_{s=2j} = \frac{-128-579j}{200} \\
&= -0.64 - j2.895 \\
C_3 &= C_2^* = -0.64 + j2.895
\end{aligned}$$

These results can also be verified with the MATLAB `residue` command,

```
a =[1 1 8 8 16 16];
b =[2 24 90 100];
[r,p,k]=residue(b,a)
```

r =
-0.64000000000000 - 2.89500000000002i
-4.15000000000002 - 1.95000000000000i
-0.64000000000000 + 2.89500000000002i
-4.15000000000002 + 1.95000000000000i
1.28000000000001
p =
0.00000000000000 + 2.00000000000000i
0.00000000000000 + 2.00000000000000i
0.00000000000000 - 2.00000000000000i
0.00000000000000 - 2.00000000000000i
-1.00000000000000
k =
[]
We then have,

$$\begin{aligned}
f(t) &= 1.28e^{-t} + 2|C_2|\cos(2t + \arg C_2) + 2|C_4|t\cos(2t + \arg C_4) \\
&= 1.28e^{-t} + 5.92979\cos(2t - 1.788) + 9.1706t\cos(2t - 2.702)
\end{aligned}$$

where
$|C_2| =$ 2.96489, $|C_4| =$ 4.5853, $\arg C_2 = \tan^{-1}(\frac{-2.895}{-0.64}) = -1.788$, using the `atan2` command in MATLAB, and $\arg C_4 = \tan^{-1}(\frac{-1.950}{-4.150}) = -2.702$ also using the `atan2` command in MATLAB.

(f)

$$F(s) = \frac{(s^2-1)}{(s^2+1)^2}$$

Using the multiplication by time Laplace transform property (Table A.1 entry #11):

$$-\frac{d}{ds}G(s) = \mathcal{L}\{tg(t)\}$$

We can see that $-\frac{d}{ds}\dfrac{s}{(s^2+1)} = \dfrac{s^2-1}{(s^2+1)^2}$.
So the inverse Laplace transform of $F(s)$ is:

$$\mathcal{L}^{-1}\{F(s)\} = t\cos t$$

(g) Follows from Problem 5 (c), or expand in series,

$$\tan^{-1}(\frac{1}{s}) = \frac{1}{s} - \frac{1}{3s^3} + \frac{1}{5s^5} - ...$$

Then,

$$\mathcal{L}^{-1}\{\tan^{-1}(\frac{1}{s})\} = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - ..... = \frac{\sin(t)}{t}.$$

Alternatively, let us assume $\mathcal{L}^{-1}\{\tan^{-1}(\frac{1}{s})) = f(t)$. We use the identity

$$\frac{d}{ds}[\tan^{-1} s] = \frac{1}{1+s^2}$$

which means that $\mathcal{L}^{-1}\{-\frac{1}{s^2+1}\} = -tf(t) = -\sin(t)$. Therefore, $f(t) = \frac{\sin(t)}{t}$.

9. Solve the following ordinary differential equations using Laplace transforms:

(a) $\ddot{y}(t) + \dot{y}(t) + 3y(t) = 0; y(0) = 1,\ \dot{y}(0) = 2$

(b) $\ddot{y}(t) - 2\dot{y}(t) + 4y(t) = 0; y(0) = 1,\ \dot{y}(0) = 2$

(c) $\ddot{y}(t) + \dot{y}(t) = \sin t; y(0) = 1,\ \dot{y}(0) = 2$

(d) $\ddot{y}(t) + 3y(t) = \sin t; y(0) = 1,\ \dot{y}(0) = 2$

(e) $\ddot{y}(t) + 2\dot{y}(t) = e^t; y(0) = 1,\ \dot{y}(0) = 2$

(f) $\ddot{y}(t) + y(t) = t; y(0) = 1,\ \dot{y}(0) = -1$

Solution:

(a)

$$\ddot{y}(t) + \dot{y}(t) + 3y(t) = 0;\ y\,(0) = 1,\ \dot{y}\,(0) = 2$$

Using Table A.1 entry #5, the differentiation Laplace transform property,

$$s^2Y(s) - sy(0) - \dot{y}(0) + sY(s) - y(0) + 3Y(s) = 0$$

$$\begin{aligned}
Y(s) &= \frac{s+3}{s^2+s+3} \\
&= \frac{\left(s+\frac{1}{2}\right)+\frac{5}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{11}{4}} \\
&= \frac{\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2+\frac{11}{4}} + \frac{5\sqrt{11}}{11}\frac{\sqrt{\frac{11}{4}}}{\left(s+\frac{1}{2}\right)^2+\frac{11}{4}}
\end{aligned}$$

Using Table A.2 entries #19 and #20,

$$y(t) = e^{-\frac{1}{2}t}\cos\frac{\sqrt{11}}{2}t + \frac{5\sqrt{11}}{11}e^{-\frac{1}{2}t}\sin\frac{\sqrt{11}}{2}t$$

(b)

$$\ddot{y}(t) - 2\dot{y}(t) + 4y(t) = 0; y(0) = 1, \dot{y}(0) = 2$$

$$s^2Y(s) - sy(0) - \dot{y}(0) - 2sY(s) + 2y(0) + 4Y(s) = 0$$

$$\begin{aligned}
Y(s) &= \frac{s+4}{s^2-2s+4} \\
&= \frac{s+4}{(s-1)^2+3} \\
&= \frac{(s-1)}{(s-1)^2+3} + \frac{5}{(s-1)^2+3} \\
&= \frac{(s-1)}{(s-1)^2+3} + \frac{5\sqrt{3}}{3}\frac{\sqrt{3}}{(s-1)^2+3}
\end{aligned}$$

Using Table A.2 entries #19 and #20,

$$y(t) = e^t\cos\sqrt{3}t + \frac{5\sqrt{3}}{3}e^t\sin\sqrt{3}t$$

(c)

$$\ddot{y}(t) + \dot{y}(t) = \sin t; y(0) = 1, \dot{y}(0) = 2$$

$$s^2Y(s) - sy(0) - \dot{y}(0) + sY(s) - y(0) = \frac{1}{s^2+1}$$

$$\begin{aligned}
Y(s) &= \frac{s^3+3s^2+s+4}{s(s+1)(s^2+1)} \\
&= \frac{C_1}{s} + \frac{C_2}{s+1} + \frac{C_3s+C_4}{s^2+1}
\end{aligned}$$

$$\begin{aligned} C_1 &= \frac{s^3+3s^2+s+4}{(s+1)(s^2+1)}|_{s=0} = 4 \\ C_2 &= \frac{s^3+3s^2+s+4}{s(s^2+1)}|_{s=-1} = -\frac{5}{2} \end{aligned}$$

$$\begin{aligned} \frac{4}{s} + \frac{-\frac{5}{2}}{s+1} + \frac{C_3 s + C_4}{s^2+1} &= \frac{s^3+3s^2+s+4}{s(s+1)(s^2+1)} \\ s^3\left(\frac{3}{2}+C_3\right) + s^2(4+C_3+C_4) + s\left(\frac{3}{2}+C_4\right) + 4 &= s^3+3s^2+s+4 \end{aligned}$$

Match coefficients of like powers of s

$$\begin{aligned} C_4 + \frac{3}{2} &= 1 \qquad \Longrightarrow C_4 = -\frac{1}{2} \\ C_3 + \frac{3}{2} &= 1 \qquad \Longrightarrow C_3 = -\frac{1}{2} \end{aligned}$$

$$\frac{4}{s} + \frac{-\frac{5}{2}}{s+1} + \frac{-\frac{1}{2}s - \frac{1}{2}}{s^2+1} = \frac{4}{s} + \frac{-\frac{5}{2}}{s+1} - \frac{1}{2}\frac{s}{s^2+1} - \frac{1}{2}\frac{1}{s^2+1}$$

Using Table A.2 entries #2, #7, #17, and #18

$$y(t) = 4 - \frac{5}{2}e^{-t} - \frac{1}{2}\cos t - \frac{1}{2}\sin t$$

(d)

$$\ddot{y}(t) + 3y(t) = \sin t; y(0) = 1, \dot{y}(0) = 2$$

$$s^2Y(s) - sy(0) - \dot{y}(0) + 3Y(s) = \frac{1}{s^2+1}$$

$$\begin{aligned} Y(s) &= \frac{s^3+2s^2+s+3}{(s^2+3)(s^2+1)} \\ &= \frac{C_1 s + C_2}{s^2+3} + \frac{C_3 s + C_4}{s^2+1} \end{aligned}$$

$$\frac{(C_1 s + C_2)\left(s^2+1\right) + (C_3 s + C_4)\left(s^2+3\right)}{(s^2+3)(s^2+1)} = \frac{s^3+2s^2+s+3}{(s^2+3)(s^2+1)}$$

Match coefficients of like powers of s:

$$\begin{aligned} &s^3(C_1+C_3) + s^2(C_2+C_4) + s(C_1+3C_3) + (C_2+3C_4) \\ &\qquad = s^3+2s^2+s+3 \end{aligned}$$

$$\begin{aligned}
C_1 + C_3 &= 1 \quad \Longrightarrow C_1 = -C_3 \\
C_2 + C_4 &= 2 \quad \Longrightarrow C_2 = 2 - C_4 \\
C_1 + 3C_3 &= 1 \quad \Longrightarrow -C_3 + 3C_3 = 1 \quad \Longrightarrow C_3 = \frac{1}{2} \\
&\Longrightarrow C_1 = -\frac{1}{2} \\
C_2 + 3C_4 &= 3 \quad \Longrightarrow (2 - C_4) + 3C_4 = 3 \quad \Longrightarrow C_4 = \frac{1}{2} \\
&\Longrightarrow C_2 = \frac{3}{2}
\end{aligned}$$

$$\begin{aligned}
Y(s) &= \frac{-\frac{1}{2}s + \frac{3}{2}}{s^2+3} + \frac{\frac{1}{2}s + \frac{1}{2}}{s^2+1} \\
&= -\frac{1}{2}\frac{s}{s^2+3} + \frac{\sqrt{3}}{2}\frac{\sqrt{3}}{s^2+3} + \frac{1}{2}\frac{s}{s^2+1} + \frac{1}{2}\frac{1}{s^2+1}
\end{aligned}$$

$$y(t) = -\frac{1}{2}\cos\sqrt{3}t + \frac{\sqrt{3}}{2}\sin\sqrt{3}t + \frac{1}{2}\cos t + \frac{1}{2}\sin t$$

(e)

$$\ddot{y}(t) + 2\dot{y}(t) = e^t; y(0) = 1, \dot{y}(0) = 2$$

$$s^2Y(s) - sy(0) - \dot{y}(0) + 2sY(s) - 2y(0) = \frac{1}{s-1}$$

$$\begin{aligned}
Y(s) &= \frac{5s-4}{s(s-1)(s+2)} \\
&= \frac{C_1}{s} + \frac{C_2}{s-1} + \frac{C_3}{s+2}
\end{aligned}$$

$$\begin{aligned}
C_1 &= \frac{5s-4}{(s-1)(s+2)}|_{s=0} = 2 \\
C_2 &= \frac{5s-4}{s(s+2)}|_{s=1} = \frac{1}{3} \\
C_3 &= \frac{5s-4}{s(s-1)}|_{s=-2} = -\frac{7}{3}
\end{aligned}$$

$$Y(s) = \frac{2}{s} + \frac{1}{3}\frac{1}{s-1} - \frac{7}{3}\frac{1}{s+2}$$

$$y(t) = 2 + \frac{1}{3}e^t - \frac{7}{3}e^{-2t}$$

(f) Using the results from Appendix A,

$$\ddot{y}(t) + y(t) = t; y(0) = 1, \dot{y}(0) = -1$$

$$s^2 Y(s) - sy(0) - \dot{y}(0) + Y(s) = \frac{1}{s^2}$$

$$\begin{aligned} Y(s) &= \frac{s^3 - s^2 + 1}{s^2(s^2+1)} \\ &= \frac{C_1}{s} + \frac{C_2}{s^2} + \frac{C_3 s + C_4}{s^2+1} \end{aligned}$$

$$\begin{aligned} C_1 &= \frac{d}{ds}\frac{\left(s^3 - s^2 + 1\right)}{(s^2+1)}|_{s=0} = 0 \\ C_2 &= \frac{\left(s^3 - s^2 + 1\right)}{(s^2+1)}|_{s=0} = 1 \end{aligned}$$

$$\begin{aligned} \frac{1}{s^2} + \frac{C_3 s + C_4}{s^2+1} &= \frac{s^3 - s^2 + 1}{s^2(s^2+1)} \\ \frac{\left(s^2+1\right) + (C_3 s + C_4)s^2}{s^2(s^2+1)} &= \frac{s^3 - s^2 + 1}{s^2(s^2+1)} \end{aligned}$$

Match coefficients of like powers of s:

$$\begin{aligned} C_3 &= 1 \\ C_4 + 1 &= -1 \qquad \Longrightarrow C_4 = -2 \end{aligned}$$

$$Y(s) = \frac{1}{s^2} + \frac{s}{s^2+1} - 2\frac{1}{s^2+1}$$

$$y(t) = t + \cos t - 2\sin t$$

10. Write the dynamic equations describing the circuit in Fig. 3.50. Write the equations in both state-variable form and as a second-order differential equation in $y(t)$. Assuming a zero input, solve the differential equation for $y(t)$ using Laplace-transform methods for the parameter values and initial conditions shown in the figure. Verify your answer using the `initial` command in MATLAB.

    Solution:

$$i = C\frac{dy}{dt} \qquad (1)$$

$$v = L\frac{di}{dt} \qquad (2)$$

$$u(t) - L\frac{di}{dt} - Ri(t) - y(t) = 0$$

$$\frac{di}{dt} = \frac{u}{L} - \frac{R}{L}i - \frac{1}{C}y \qquad (3)$$

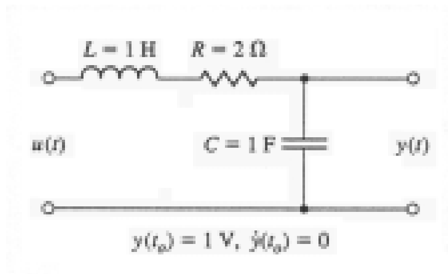


Figure 3.50: Circuit for Problem 3.10

Put differential equations (2) and (3) into state-space form:

$$\begin{bmatrix} \dot{i} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

Substituting the given values for L, R, and C we have for equation (3):

$$\frac{di}{dt} = u - 2\frac{dy}{dt} - y$$
$$\ddot{y} + 2\dot{y} + y = u$$

Characteristic equation:

$$\begin{aligned} s^2 + 2s + 1 &= 0 \\ (s+1)^2 &= 0 \end{aligned}$$

So:

$$y(t) = A_1 e^{-t} + A_2 t e^{-t}$$

Solving for the coefficients:

$$\begin{aligned} y(t) &= A_1 e^{-t} + A_2 t e^{-t} \\ y(t_0) &= A_1 e^{-t_0} + A_2 t_0 e^{t_0} = 1 \\ \dot{y}(t) &= -A_1 e^{-t} + A_2 e^{-t} - A_2 t e^{-t} \\ \dot{y}(t_0) &= -A_1 e^{-t_0} + A_2 e^{-t_0} - A_2 t e^{-t_0} = 0 \end{aligned}$$

$$\Rightarrow A_2 = e^{t_0} \quad \text{and} \quad A_1 = (1 - t_0)e^{t_0}$$
$$y(t) = (1 - t_0)e^{t_0 - t} + t e^{t_0 - t}$$

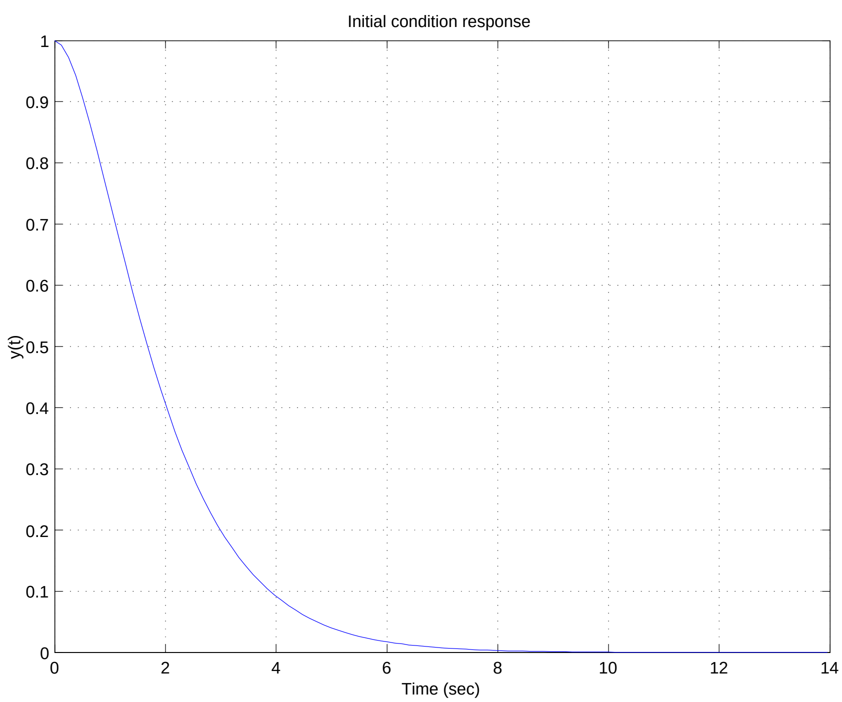
To verify the solution using MATLAB, re-write the differential equation as,

$$\begin{aligned} \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \end{aligned}$$

Then the following MATLAB statements,

```
a=[0,1;-1,-2];
b=[0;1];
c=[1,0];
d=[0];
sys=ss(a,b,c,d);
xo=[1;0];
[y,t,x]=initial(sys,xo);
plot(t,y);
grid;
xlabel('Time (sec)');
ylabel('y(t)');
title('Initial condition response');
```

generate the initial condition response shown below that agrees with the analytical solution above.



Problem 3.10: Initial condition response.

11. Consider the standard second-order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

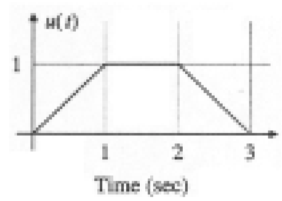


Fig.3.51

Figure 3.51: Plot of input for Problem 3.11

a) Write the Laplace transform of the signal in Fig. 3.51. b). What is the transform of the output if this signal is applied to G(s). c) Find the output of the system for the input shown in Fig. 3.51.

Solution:

(a) The input signal in Figure 3.51 may be written as:

$$u(t) = t - t[1(t-1)] - t[1((t-2)] + t[1(t-3)]$$

where $1(t-\tau)$ denotes a unit step.
The Laplace transform of the input signal is:

$$U(s) = \frac{1}{s^2}\left(1 - e^{-s} - e^{-2s} - e^{-3s}\right)$$

(b) The Laplace transform of the output if this signal is applied is:

$$Y(s) = G(s)U(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\left(\frac{1}{s^2}\right)\left(1 - e^{-s} - e^{-2s} - e^{-3s}\right)$$

(c) However to make the mathematical manipulation easier, consider only the reponse of the system to a ramp input:

$$Y_1(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\left(\frac{1}{s^2}\right)$$

Partial fractions yields the following:

$$Y_1(s) = \frac{1}{s^2} - \frac{\frac{2\zeta}{\omega_n}}{s} + \frac{\frac{2\zeta}{\omega_n}(s + 2\zeta\omega_n - \frac{\omega_n}{2\zeta})}{(s + \omega_n\zeta)^2 + (\omega_n\sqrt{1-\zeta^2})^2}$$

Use the following Laplace transform pairs for the case $0 \le \zeta < 1$:

$$\mathcal{L}^{-1}\{\frac{s+z_1}{(s+a)^2+\omega^2}\} = \sqrt{\frac{(z_1-a)^2+\omega^2}{\omega^2}}e^{-at}\sin(\omega t+\phi)$$

where $\phi \equiv \tan^{-1}\left(\frac{\omega}{z_1-a}\right)$

$$\mathcal{L}^{-1}\{\frac{1}{s^2}\} = t \qquad \text{ramp}$$

$$\mathcal{L}^{-1}\{\frac{1}{s}\} = 1(t) \qquad \text{unit step}$$

and the following Laplace transform pairs for the case $\zeta = 1$:

$$\mathcal{L}^{-1}\{\frac{1}{(s+a)^2}\} = te^{-at}$$

$$\mathcal{L}^{-1}\{\frac{s}{(s+a)^2}\} = (1-at)e^{-at}$$

$$\mathcal{L}^{-1}\{\frac{1}{s^2}\} = t \qquad \text{ramp}$$

$$\mathcal{L}^{-1}\{\frac{1}{s}\} = 1(t) \qquad \text{unit step}$$

the following is derived:

$$y_1(t) = \left\{ \begin{array}{cc} t - \frac{2\zeta}{\omega_n} + \frac{e^{-\zeta\omega_n t}}{\omega_n\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2}t + \tan^{-1}\frac{2\zeta\sqrt{1-\zeta^2}}{2\zeta^2-1} & \begin{array}{c} 0 \le \zeta < 1 \\ t \ge 0 \end{array} \\ t - \frac{2}{\omega_n} + \frac{2}{\omega_n}e^{-\omega_n t}\left(\frac{\omega_n}{2}t+1\right) & \begin{array}{c} \zeta = 1 \\ t \ge 0 \end{array} \end{array} \right\}$$

Since $u(t)$ consists of a ramp and three delayed ramp signals, using superpostion (the system is linear), then:

$$y(t) = y_1(t) - y_1(t-1) - y_1(t-2) + y_1(t-3) \qquad t \ge 0$$

12. A rotating load is connected to a field-controlled DC motor with negligible field inductance. A test results in the output load reaching a speed of 1 rad/sec within 1/2 sec when a constant input of 100 V is applied to the motor terminals. The output steady-state speed from the same test is found to be 2 rad/sec. Determine the transfer function $\theta(s)/V_f(s)$ of the motor.

Solution:

Equations of motion for a DC motor:

$$J_m\ddot{\theta}_m + b\dot{\theta}_m = K_m i_a,$$

$$K_e\dot{\theta}_m + L_a\frac{di_a}{dt} + R_a i_a = v_a,$$

but since there's neglible field inductance $L_a = 0$.
Combining the above equations yields:

$$R_a J_m \ddot{\theta}_m + R_a b\dot{\theta}_m = K_t v_a - K_t K_e \dot{\theta}_m$$

Applying Laplace transforms yields the following transfer function:

$$\frac{\theta(s)}{V_f(s)} = \frac{\frac{K_t}{J_m R_a}}{s(s + \frac{K_t K_e}{R_a J_m} + \frac{b}{J_m})} = \frac{K}{s(s+a)}$$

where $K = \frac{K_t}{J_m R_a}$ and $a = \frac{K_t K_e}{R_a J_m} + \frac{b}{J_m}$.
K and a are found using the given information:

$$\begin{aligned} V_f(s) &= \frac{100}{s} \text{ since } V_f(t) = 100V \\ \dot{\theta}(\frac{1}{2}) &= 2 \text{ rad/sec} \end{aligned}$$

For the given information we need to utilize $\dot{\theta}_m(t)$ instead of $\theta_m(t)$:

$$s\theta(s) = \frac{100K}{s(s+a)}$$

Using the Final Value Theorem and assuming that the system is stable:

$$\lim_{s\to 0}\frac{100K}{s+a} = \lim_{s\to 0}\dot{\theta}(\frac{1}{2}) = 2 = \frac{100K}{a}$$

Take the inverse Laplace transform:

$$\begin{aligned} \mathcal{L}^{-1}\{\frac{100K}{a}\frac{a}{s(s+a)}\} &= \frac{100K}{a}(1 - e^{-at}) = 2(1 - e^{-at}) = 1 \\ 0.5 &= e^{-\frac{a}{2}} \text{ yields } a = 1.39 \\ K &= \frac{2}{100}a \text{ yields } K = 0.0278 \\ \frac{\theta(s)}{V_f(s)} &= \frac{0.0278}{s(s+1.39)} \end{aligned}$$

13. For the tape drive shown in Fig. 2.48, compute the following, using the numbers given in Problem 2.20 (a):

(a) the transfer function from the motor current to the tape position;

(b) the poles and zeros for the transfer function in part (a).

Solution:

(a)

$$\frac{Tape\_tension}{I_a(s)} = \frac{T(s)}{I_a(s)}$$

$$\begin{aligned} T &= B(\dot{x}_2 - \dot{x}_1) + k(x_2 - x_1) \\ T(s) &= (Bs + k)(X_2(s) - X_1(s)) \end{aligned}$$

From Problem 2.20:

$$\begin{aligned} J_1\dot{\omega}_1 &= -B_1\omega_1 + k_t i_a + Br_1(\dot{x}_2 - \dot{x}_1) + kr_1(x_2 - x_1) \\ J_2\dot{\omega}_2 &= -B_2\omega_2 + Fr_2 + Br_2(\dot{x}_1 - \dot{x}_2) + kr_2(x_1 - x_2) \\ \dot{x}_1 &= r_1\omega_1 \\ \dot{x}_2 &= r_2\omega_2 \end{aligned}$$

$$\begin{bmatrix} J_1 s + B_1 & 0 & (Bs+k)r_1 & -(Bs+k)r_1 \\ 0 & J_2 s + B_2 & -(Bs+k)r_2 & (Bs+k)r_2 \\ -r_1 & 0 & s & 0 \\ 0 & -r_2 & 0 & s \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k_t I_a \\ Fr_2 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} X_1(s) &= I_a(s)\frac{15(s+800)}{s(s^2 + 1250s + 0.4 \times 10^6)} \\ X_2(s) &= I_a(s)\frac{4.8 \times 10^{-6}}{s(s^2 + 1250s + 0.4 \times 10^6)} \\ X_2(s) - X_1(s) &= \frac{15(s+800) - 4.8 \times 10^{-6}}{s(s^2 + 1250s + 0.4 \times 10^6)} I_a(s) \\ \frac{T(s)}{I_a(s)} &= (20s + 2 \times 10^4)\frac{15(s+800) - 4.8 \times 10^{-6}}{s(s^2 + 1250s + 0.4 \times 10^6)} \end{aligned}$$

(b)

$$\begin{aligned} \text{poles at} &: \quad 0, -625 \pm j996.8 \\ \text{zeros at} &: \quad -800, -1000 \end{aligned}$$

14. For the system in Fig. 2.50, compute the transfer function from the motor voltage to position θ_2.

Solution:

From Problem 2.23:

$$\begin{aligned} L\frac{di_a}{dt} + R_a i_a + k_e\dot{\theta}_1 &= v_a \\ k_t i_a &= J_1\ddot{\theta}_1 + b(\dot{\theta}_1 - \dot{\theta}_2) + k(\theta_1 - \theta_2) + B\dot{\theta}_1 \\ J_2\ddot{\theta}_2 + b(\dot{\theta}_2 - \dot{\theta}_1) + k(\theta_2 - \theta_1) &= 0 \end{aligned}$$

So we have:

$$\begin{aligned} LsI_a(s) + R_a I_a(s) + sk_e\Theta_1(s) &= V_a(s) \\ k_t I_a(s) &= s^2 J_1\Theta_1(s) + b[\Theta_1(s) - \Theta_2(s)]s + k[\Theta_1(s) - \Theta_2(s)] + Bs\Theta_1(s) \\ s^2 J_2\Theta_2(s) + b[\Theta_2(s) - \Theta_1(s)]s + k[\Theta_2(s) - \Theta_1(s)] &= 0 \end{aligned}$$

we have:

$$\begin{aligned} \frac{\Theta_2(s)}{V_a(s)} &= \frac{k_t(bs+k)}{\det\begin{bmatrix} sk_e & 0 & Ls+R_a \\ J_1s^2+Bs+bs+k & -bs-k & -k_t \\ -bs-k & J_2s^2+bs+k & 0 \end{bmatrix}} \\ &= \frac{k_t(bs+k)}{\begin{array}{c}(Ls+Ra)[J_1J_2s^4 + (J_1b+BJ_2+bJ_2)s^3 + (J_1k+Bb+KJ_2)s^2 + Bks] \\ +k_ek_tJ_2s^3 + k_ek_tbs^2 + kk_ek_ts\end{array}} \\ &= \frac{k_t(bs+k)}{\begin{array}{c} J_1J_2s^5 + J_2[J_1R_a + L(b+B)]s^4 \\ +[J_2k_ek_tJ_1L(b+k) + LJ_2k + R_a(b+B)J_2 - Lb^2]s^3 \\ +[L(b+B)(b+k) - 2bkL + J_1R_a(b+k) + R_aJ_2k - b^2R_a]s^2 \\ +[k_ek_t(b+k) + kL(b+k) - bk^2 + R_a(b+B)(b+k) - 2bkR_a]s + kR_ab \end{array}} \end{aligned}$$

15. Compute the transfer function for the two-tank system in Fig. 2.54 with holes at A and C.

Solution:

From Problem 2.27 but with $s = a$ tank area we have:

$$\begin{bmatrix} \Delta\dot{h}_1 \\ \Delta\dot{h}_2 \end{bmatrix} = \frac{1}{6a}\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \Delta h_1 \\ \Delta h_2 \end{bmatrix} + \frac{\omega_{in}}{a}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{-10}{3a} \\ 0 \end{bmatrix}$$

$$\begin{aligned}
\Delta\dot{h}_1 &= \frac{-\Delta h_1 + 6\omega_{in} - 20}{6a} \\
\Delta\dot{h}_2 &= \frac{1}{6a}(\Delta h_1 - \Delta h_2) \\
s\Delta h_1(s) &= \frac{-\Delta h_1(s) + 6\omega_{in}(s)}{6a} \\
s\Delta h_2(s) &= \frac{1}{6a}[\Delta h_1(s) - \Delta h_2(s)] \\
\Delta h_2(s) &= \frac{\omega_{in}(s)}{6a[a(\frac{1}{6a}+s)]^2} \\
\frac{\Delta h_2(s)}{\omega_{in}(s)} &= \frac{1}{6[a(\frac{1}{6a}+s)]^2}
\end{aligned}$$

16. For a second-order system with transfer function

$$G(s) = \frac{3}{s^2 + 2s - 3},$$

determine the following:

(a) DC gain;

(b) the final value to a step input.

Solution:

(a) DC gain $G(0) = \frac{3}{-3} = -1$

(b) $\lim_{t\to\infty} y(t) = ?$
$s^2 + 2s + 3 = 0 \Longrightarrow s = 1, -3$
Since the system has an unstable pole, the Final Value Theorem is not applicable. The output is unbounded.

17. Consider the continuous rolling mill depicted in Fig. 3.52. Suppose that the motion of the adjustable roller has a damping coefficient b, and that the force exerted the rolled material on the adjustable roller is proportional to the material's change in thickness: $F_s = c(T - x)$. Suppose further that the DC motor has a torque constant K_t and a back-emf constant K_e, and that the rack-and-pinion has effective radius of R.

(a) What are the inputs to this system? The output?

(b) Without neglecting the effects of gravity on the adjustable roller, draw a block diagram of the system that explicitly shows the following quantities: $V_s(s)$, $I_0(s), F(s)$ (the force the motor exerts on the adjustable roller), and $X(s)$.

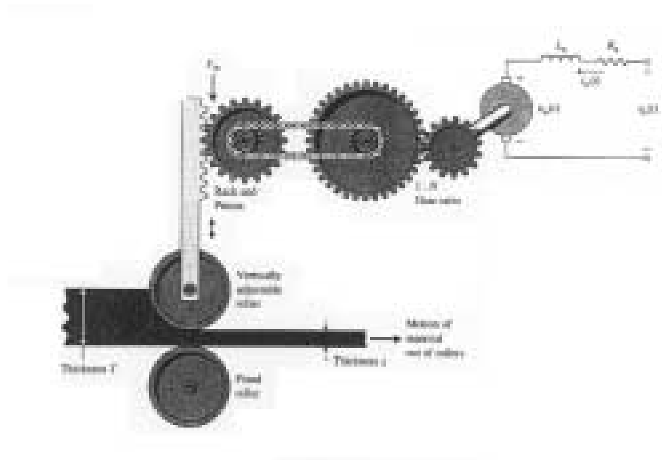


Figure 3.52: Continuous rolling mill

(c) Simplify your block diagram as much as possible while still identifying output and each input separately.

Solution:

(a)

<u>Inputs</u>:	input voltage $\longrightarrow v_s(t)$
	thickness $\longrightarrow$ T
	gravity $\longrightarrow$ mg
<u>Output</u>:	thickness $\longrightarrow x$

(b) Dynamic analysis of adjustable roller:

$$m\ddot{x} = c(T - x) - mg - b\dot{x} - F_m$$
$$\Longrightarrow (s^2m + sb + c)X(s) + F_m(s) + \frac{mg - cT}{s} = 0 \qquad (1)$$

Torque in rack and pinion:

$$T_{RP} = RF_m = NT_{motor}$$
$$\text{but } T_{motor} = K_t I_f i_o$$
$$F_m = \frac{NK_tI_f}{R} i_o \qquad (2)$$

DC motor circuit analysis:

$$v_s(t) = R_a i_o + L_a \frac{di_o}{dt} + v_a(t)$$

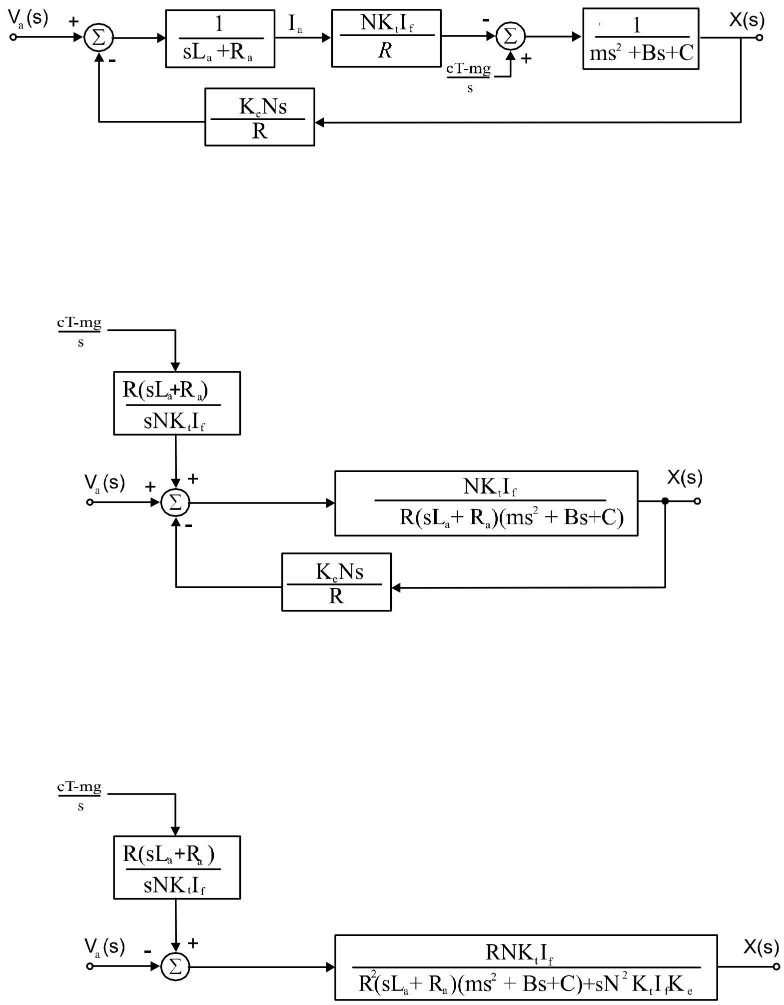
$$v_a(t) = u_e \dot{\theta}$$

$$\frac{\theta R}{N} = x$$

$$I_o(s) = \frac{V_s(s) - \frac{K_e N}{R} sX(s)}{R_a + sL_a} \qquad (3)$$

Combining (1), (2), and (3):

$$0 = (s^2 m + sb + c)X(s) + \frac{mg - cT}{s} + \frac{NK_t I_f}{R} \left[\frac{V_s(s) - \frac{K_e N}{R} sX(s)}{sL_a + R_a} \right]$$



Block diagrams for rolling mill

Problems and Solutions for Section 3.2

18. Compute the transfer function for the block diagram shown in Fig. 3.53. Note that a_i and b_i are constants.

    (a) Write the third-order differential equation that relates y and u. (**Hint**: Consider the transfer function.)

    (b) Write three simultaneous first-order (state-variable) differential equations using variables x_1, x_2, and x_3, as defined on the block diagram in Fig. 3.53. Notice how the same constant parameters enter the transfer function, the differential equations, and the matrices of the state-variable form. (This special structure is called the control

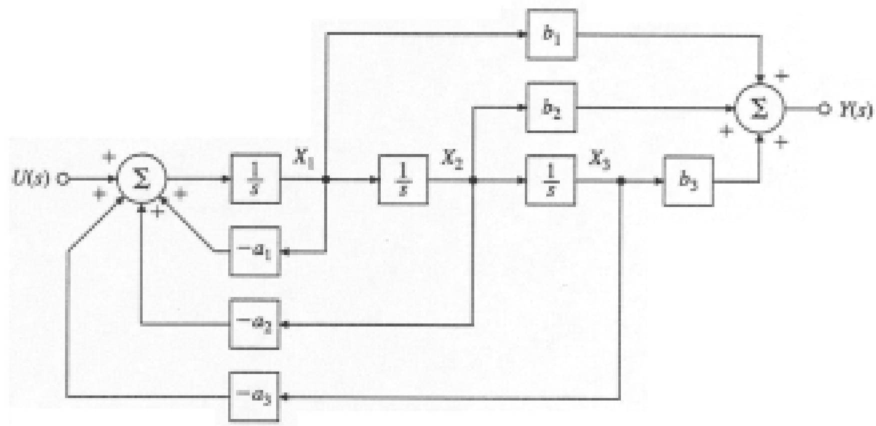


Figure 3.53: Block diagram for Problem 3.18

canonical form and will be discussed further in Chapter 7.) Repeat for the block diagram of Fig. 3.50(b). This is the "observer canonical form" for a 3rd order system.

Solution:

Using Mason's rule:

Forward paths:

$$\frac{b_1}{s} + \frac{b_2}{s^2} + \frac{b_3}{s^3}$$

Feedback paths:

$$\frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3}$$

$$\frac{Y}{U} = \frac{\frac{b_1}{s} + \frac{b_2}{s^2} + \frac{b_3}{s^3}}{1 + \frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3}}$$

(a)

$$\begin{aligned}(s^3 + a_1 s^2 + a_2 s + a_3)Y &= (b_1 s^2 + b_2 s + b_3)U \\ \frac{d^3 y}{dt^3} + a_1 \ddot{y} + a_2 \dot{y} + a_3 y &= b_1 \ddot{u} + b_2 \dot{u} + b_3 u\end{aligned}$$

(b) Definitions from block diagram:

$$\begin{aligned}\dot{x}_3 &= x_2 \\ \dot{x}_2 &= x_1 \\ \dot{x}_1 &= u - a_1 x_1 - a_2 x_2 - a_3 x_3 \\ y &= b_3 x_3 + b_2 x_2 + b_1 x_1\end{aligned}$$

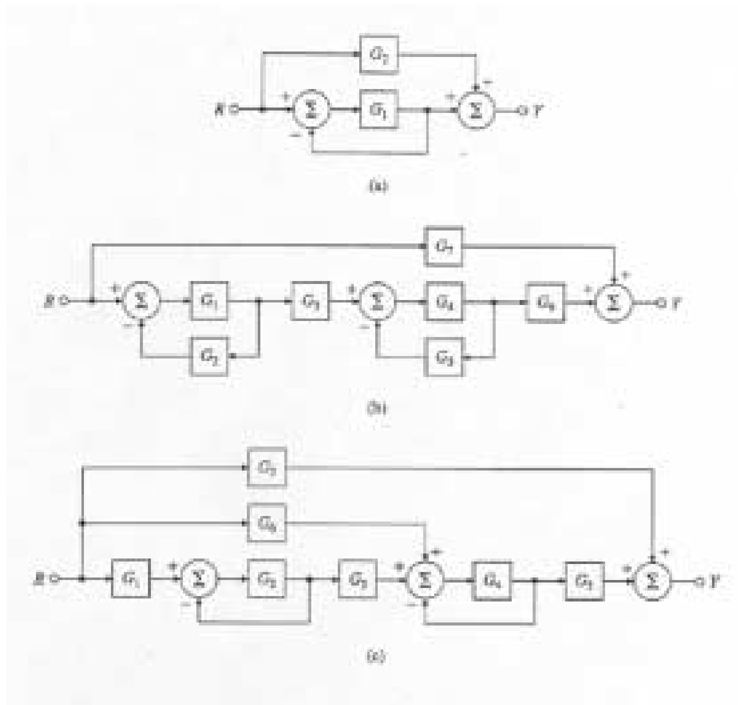


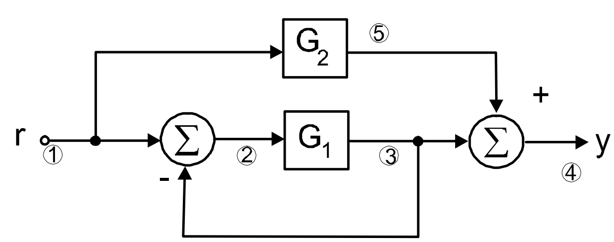
Figure 3.54: Block diagrams for Problem 3.19

$$\dot{\mathsf{x}} = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix} \mathsf{x}$$

19. Find the transfer functions for the block diagrams in Fig. 3.54.

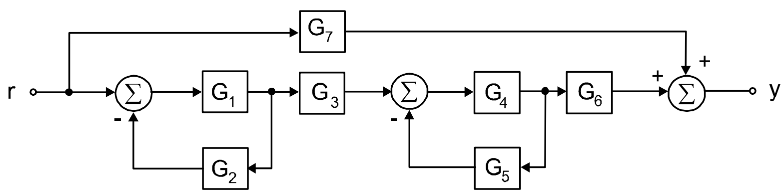
Solution:



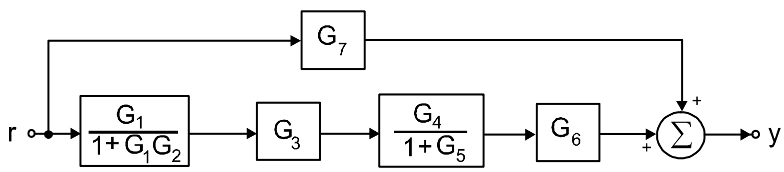
(a) Block diagram for Fig. 3.54 (a)

Forward Path	Gain
1 2 3 4	G_1
1 5 4	G_2
Loop Path	**Gain**
2 3 2	$-G_1$

$$\frac{Y}{R} = \frac{G_1}{1+G_1} + G_2$$

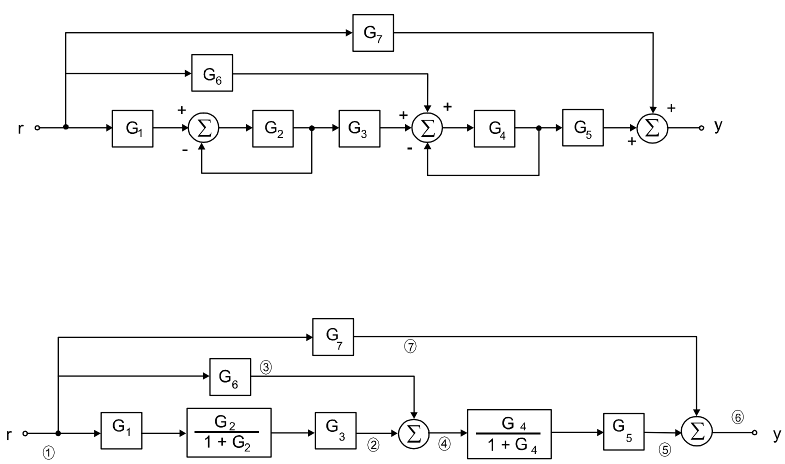


(b) Block diagram for Fig. 3.54 (b)



Block diagram for Fig. 3.54 (b): reduced

$$\frac{Y}{R} = G_7 + \frac{G_1G_3G_4G_6}{(1+G_1G_2)(1+G_4G_5)}$$



(c) Block diagrams for Fig. 3. 54(c)

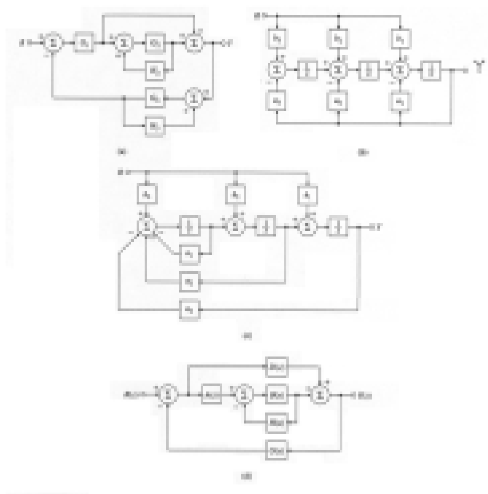


Figure 3.55: Block diagrams for Problem 3.20

Forward Path	Gain
1 2 3 4 5 6	$\frac{G_1G_2G_3}{1+G_2} \times \frac{G_4G_5}{1+G_4}$
1 3 4 5 6	$\frac{G_6G_4G_5}{1+G_4}$
1 7 6	G_7

$$\frac{Y}{R} = G_7 + \frac{G_6G_4G_5}{1+G_4} + \frac{G_1G_2G_3}{1+G_2} \times \frac{G_4G_5}{1+G_4}$$

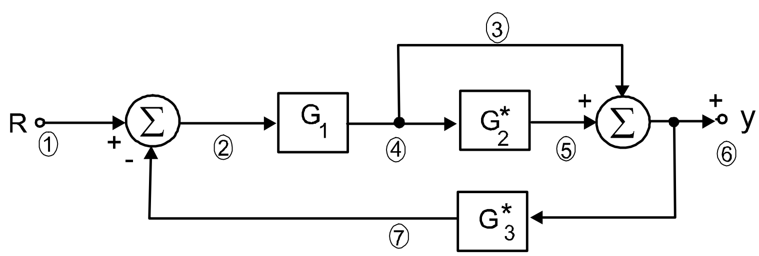
20. Find the transfer functions for the block diagrams in Fig. 3.55, using the following:

    (a) the ideas of Figs. 3.6 and 3.7;

    (b) Mason's rule.

Solution:

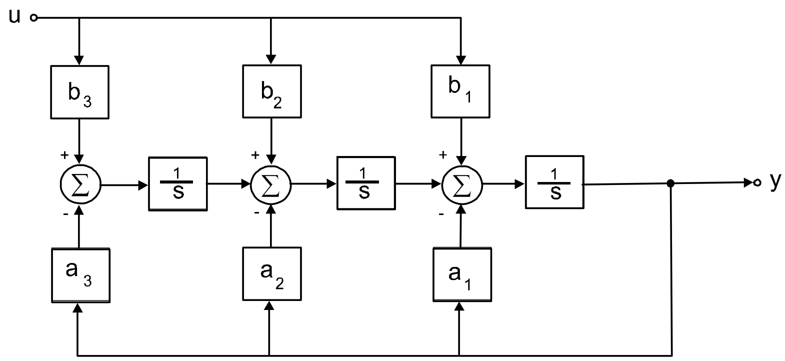
Transfer functions found using the ideas of Figs. 3.6 and 3.7:



(a) Block diagram for Fig. 3.55(a)

$$\begin{aligned} G_2^* &= \frac{G_2}{1-G_2H_2} \\ G_3^* &= \frac{G_3}{1-G_3H_3} \end{aligned}$$

$$\frac{Y}{R} = \frac{G_1(1+G_2^*)}{1+G_1(1+G_2^*)G_3^*} = \frac{G_1(1-G_2H_2)(1-G_3H_3)+G_1G_2(1-G_3H_3)}{(1-G_2H_2)(1-G_3H_3)+G_1G_3(1-G_2H_2)+G_1G_2G_3}$$

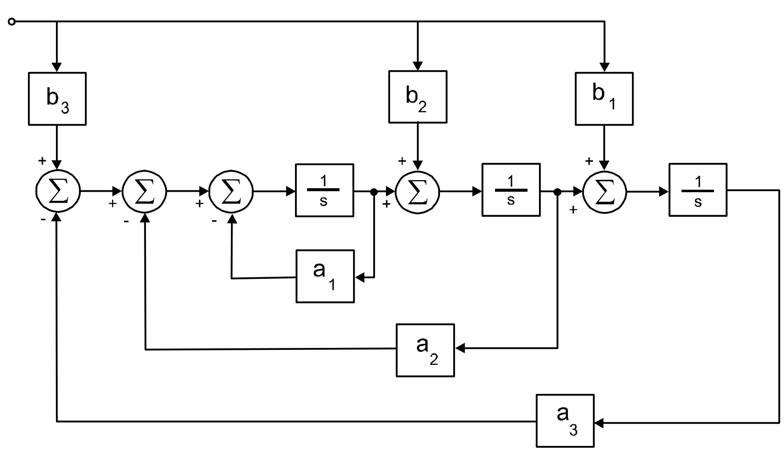


(b) Block diagram for Fig. 3.55(b)

Mason's rule: forward path gains,

$$\frac{b_3}{s^3}, \frac{b_2}{s^2}, \frac{b_1}{s}$$

(c) Applying block diagram reduction:

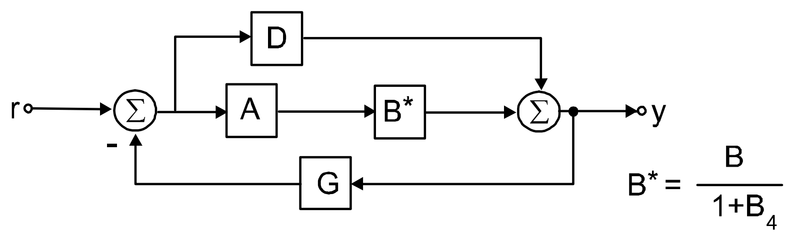


Block diagram for Fig. 3.55(c)

The method: reduce innermost box, shift b_2 to b_3 node, reduce next innermost box and continue systematically.

(d)

$$\frac{Y}{R} = \frac{b_3 + b_2(s + a_1) + b_1(s^2 + a_1 s + a_2)}{s^3 + a_1 s^2 + a_2 s + a_3}$$



Block diagram for Fig. 3.55(d)

The system is tightly connected , easy to apply Mason's.

Mason's rule:

(a)

Forward Path

1 2 3 5 6 $\quad p_1 = G_1 G_2^*$

1 2 3 4 6 $\quad p_2 = G_1$

Loop Path

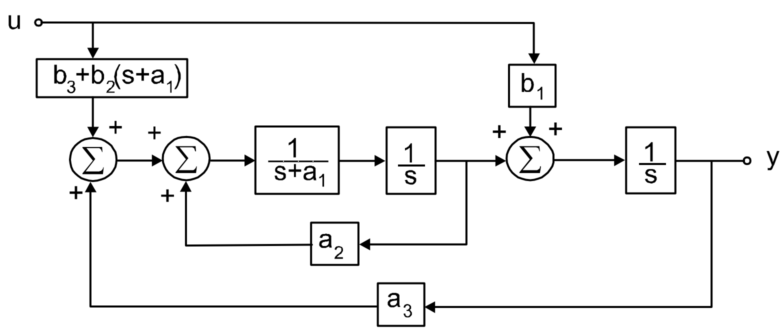
2 3 4 7 $\quad L_1 = G_1 G_3^*$

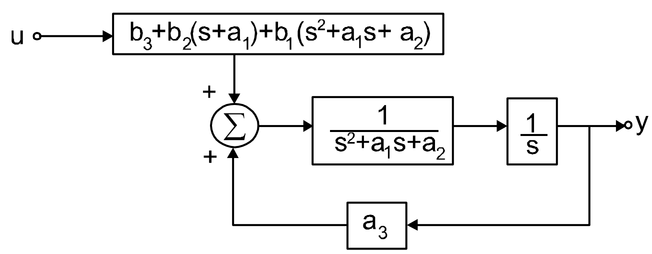
2 3 5 7 $\quad L_2 = G_1 G_2^* G_3^*$

$$\frac{Y}{R} = \frac{p_1 + p_2}{1 + L_1 + L_2} = \frac{G_1(1 + G_2^*)}{1 + G_1 G_3^*(1 + G_2^*)}$$

(b) Loop Path: $-\frac{a_3}{s^3}, -\frac{a_2}{s^2}, -\frac{a_1}{s}$

$$\frac{Y}{R} = \frac{\frac{b_3}{s^3} + \frac{b_2}{s^2} + \frac{b_1}{s}}{1 + \frac{a_3}{s^3} + \frac{a_2}{s^2} + \frac{a_1}{s}} = \frac{b_3 + b_2 s + b_1 s^2}{s^3 + a_1 s^2 + a_2 s + a_3}$$





(c) Block diagrams for Fig. 3.55(c)

(d) Mason's Rule:

$$\frac{Y}{R} = \frac{D + AB^*}{1 + G(D + AB^*)} = \frac{D + DBH + AB}{1 + BH + GD + GBDH + GAB}$$

21. Use block-diagram algebra or Mason's rule to determine the transfer function between R(s) and Y(s) in Fig. 3.56.

Solution:

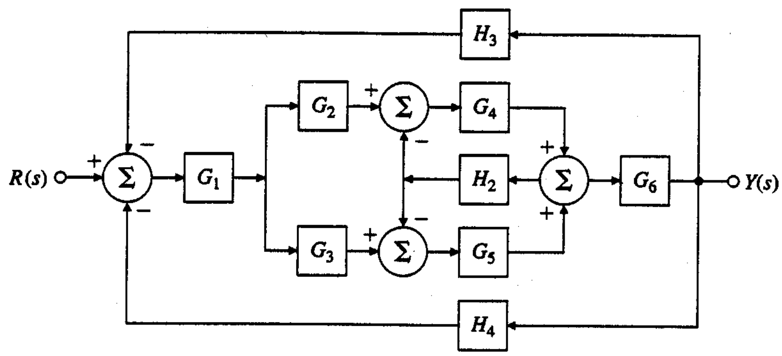
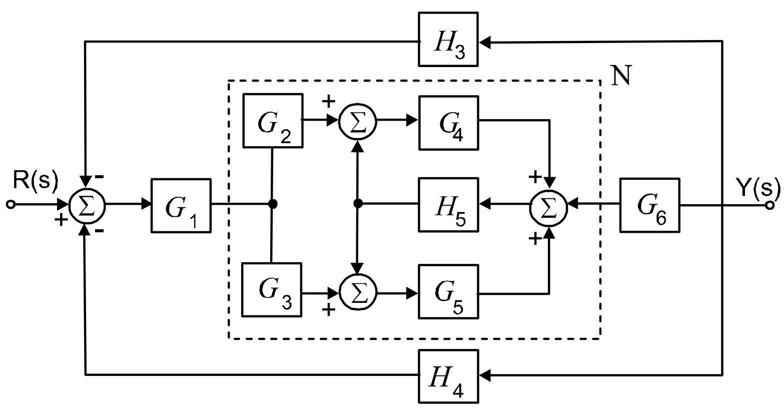
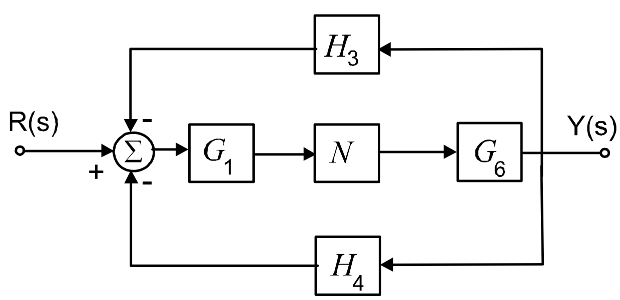


Figure 3.56: Block diagram for Problem 3.21



Block diagram for Fig. 3.56

By block diagram algebra:



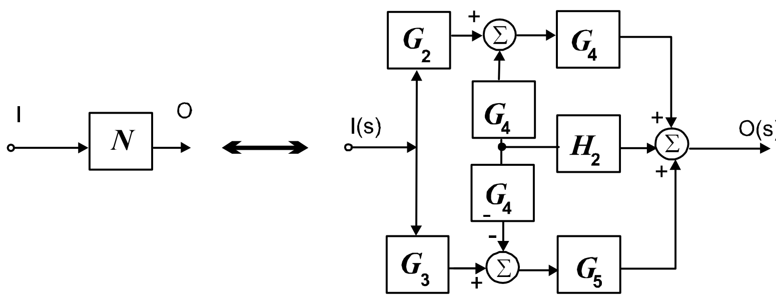
Block diagram for Fig. 3.56: reduced

$$
\begin{aligned}
Q &= R - PH_3 - PH_4 \\
&= R - P(H_3 - H_4) \\
P &= G_1 N G_6 = Y
\end{aligned}
$$

So:

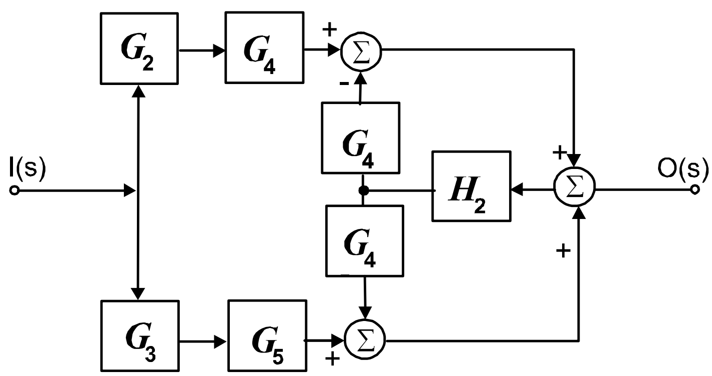
$$\frac{Y}{R} = \frac{G_1 N G_6}{1 + (H_3 + H_4) G_1 N G_6}$$

Now, what is N?



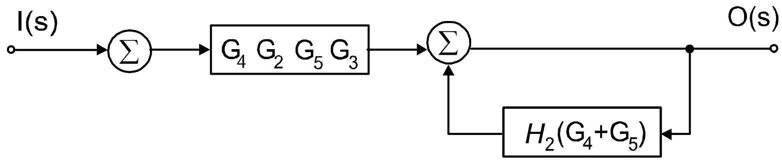
Block diagrams for Fig. 3.56

Move G_4 and G_3 :



Block diagram for Fig. 3.56: reduced

Combine symmetric loops as in the first step:



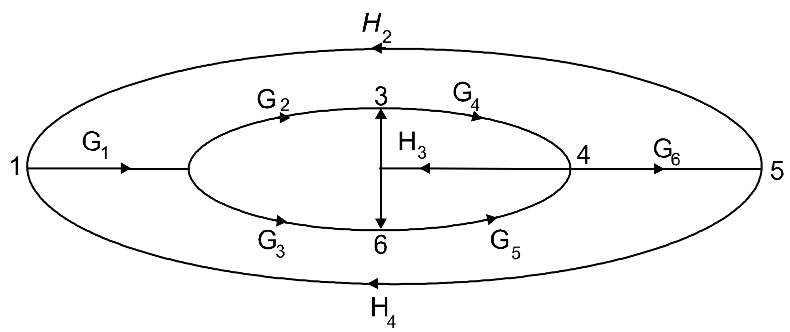
Block diagram for Fig. 3.56: reduced

Which is:

$$
\begin{aligned}
N &= \frac{O}{I} = \frac{G_4G_2 + G_5G_3}{1 + H_2(G_4 + G_5)} \\
\frac{Y(s)}{R(s)} &= \frac{G_1(G_4G_2 + G_5G_3)G_6}{1 + H_2(G_4 + G_5) + (H_3 + H_4)G_1(G_4G_2 + G_5G_3)G_6}
\end{aligned}
$$

By Mason's Rule:

Signal flow graph



Flow graph for Fig. 3.56

Forward Path	Gain
1 2 3 4 5	$G_1G_2G_4G_6$
1 2 6 4 5	$G_1G_3G_5G_6$
Loop Path	**Gain**
1 2 3 4 5 1	$-G_1G_2G_4G_6H_3$
1 2 3 4 5 1	$-G_1G_2G_4G_6H_4$
1 2 6 4 5 1	$-G_1G_3G_5G_6H_3$
1 2 6 4 5 1	$-G_1G_3G_5G_6H_4$
3 4 3	$-G_4H_2$
3 4 3	$-G_5H_2$

and the determinants are

$$
\begin{aligned}
\Delta &= 1 + [(H_3 + H_4)G_1(G_2G_4 + G_3G_5)G_6 + H_2(G_4 + G_5)] \\
\Delta_1 &= 1 - (0) \\
\Delta_2 &= 1 - (0) \\
\Delta_3 &= 1 - (0) \\
\Delta_4 &= 1 - (0)
\end{aligned}
$$

Applying the rule, the transfer function is

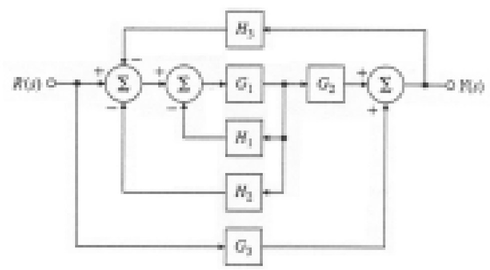
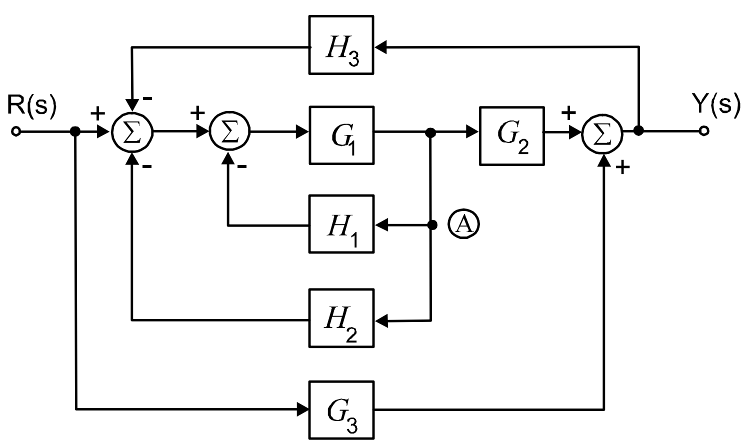


Figure 3.57: Block diagram for Problem 3.22

$$
\begin{aligned}
\frac{Y(s)}{R(s)} &= \frac{1}{\Delta}\sum G_i \Delta_i \\
&= \frac{G_1(G_4G_2 + G_5G_3)G_6}{1 + H_2(G_4 + G_5) + (H_3 + H_4)G_1(G_4G_2 + G_5G_3)G_6}
\end{aligned}
$$

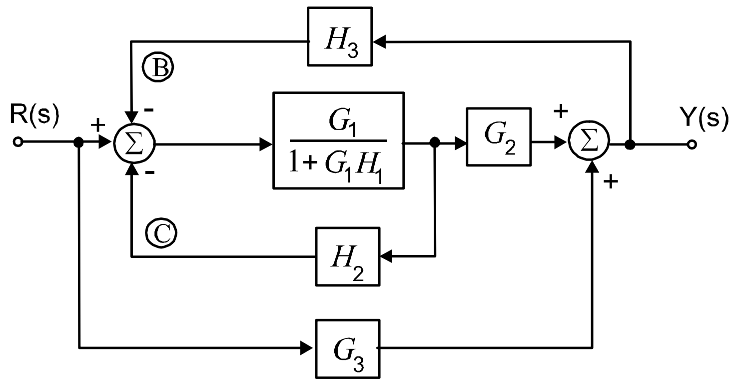
22. Use block-diagram algebra to determine the transfer function between R(s) and Y(s) in Fig. 3.57.

Solution:



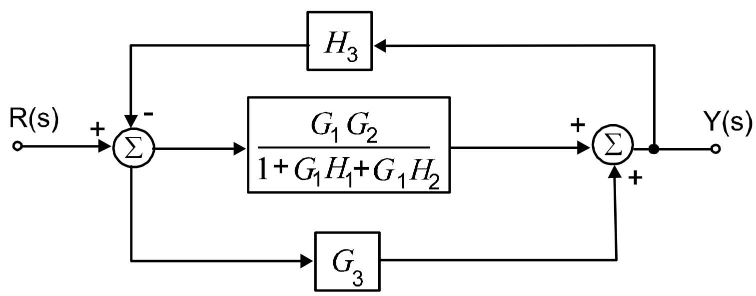
Block diagram for Fig. 3.57

Move node A and close the loop:



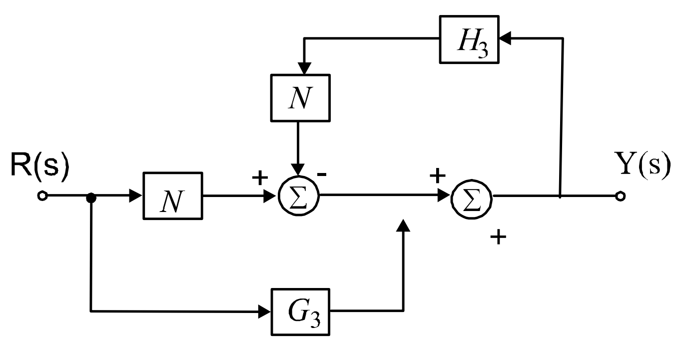
Block diagram for Fig. 3.57: reduced

Add signal B, close loop and multiply before signal C.



Block diagram for Fig. 3.57: reduced

Move middle block N past summer.



Block diagram for Fig. 3.57: reduced

Now reverse order of summers and close each block separately.

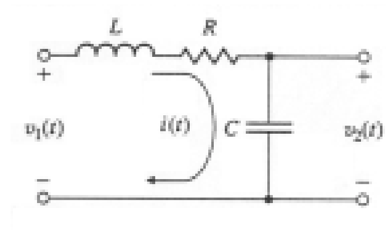
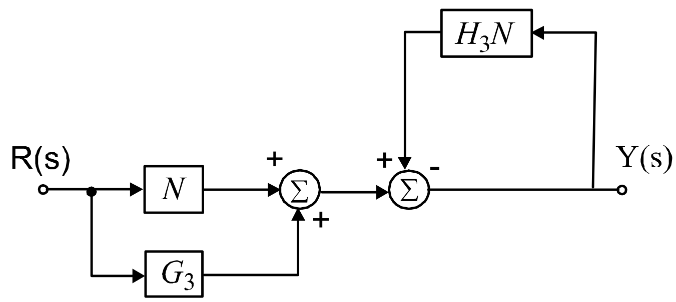


Figure 3.58: Circuit for Problem 3.23



Block diagram for Fig. 3.57: reduced

$$\frac{Y}{R} = \overbrace{(N + G_3)}^{feedforward} \underbrace{\left(\frac{1}{1 + NH_3}\right)}_{feedback}$$

$$\frac{Y}{R} = \frac{G_1G_2 + G_3(1 + G_1H_1 + G_1H_2)}{1 + G_1H_1 + G_1H_2 + G_1G_2H_3}$$

23. For the electric circuit shown in Fig. 3.58, find the following:

    (a) the time-domain equation relating i(t) and $v_1(t)$;

    (b) the time-domain equation relating i(t) and $v_2(t)$;

    (c) assuming all initial conditions are zero, the transfer function $V_2(s)/V_1(s)$ and the damping ratio ζ and undamped natural frequency ω_n of the system;

    (d) the values of R that will result in $v_2(t)$ having an overshoot of no more than 25%, assuming $v_1(t)$ is a unit step, L = 10 mH, and $C = 4\ \mu$F.

Solution:

(a)

$$v_1(t) = L\frac{di}{dt} + Ri + \frac{1}{C}\int i(t)dt$$

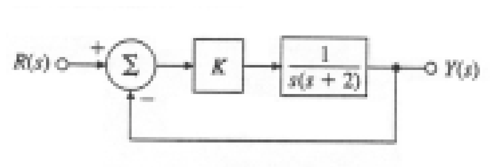


Figure 3.59: Unity feedback system for Problem 3.24

(b)

$$v_2(t) = \frac{1}{C}\int i(t)dt$$

(c)

$$\frac{v_2(s)}{v_1(s)} = \frac{\frac{1}{sC}}{sL + R + \frac{1}{sC}} = \frac{1}{s^2LC + sRC + 1}$$

(d) For 25% overshoot $\zeta \approx 0.4$

$$\begin{aligned} 0.4 &\approx \zeta = \frac{R}{2\sqrt{\frac{L}{C}}} \\ R &= 2\zeta\sqrt{\frac{L}{C}} = (2)(0.4)\sqrt{\frac{10 \times 10^{-3}}{4 \times 10^{-6}}} = 40\Omega \end{aligned}$$

Problems and Solutions for Section 3.3

24. For the unity feedback system shown in Fig. 3.59, specify the gain K of the proportional controller so that the output $y(t)$ has an overshoot of no more than 10% in response to a unit step.

Solution:

$$\begin{aligned} \frac{Y(s)}{R(s)} &= \frac{K}{s^2 + 2s + K} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ \omega_n &= \sqrt{K} \\ \zeta &= \frac{2}{2\omega_n} = \frac{1}{\sqrt{K}} \qquad (1) \end{aligned}$$

In order to have an overshoot of no more than 10%:

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \leq 0.10$$

Solving for ζ :

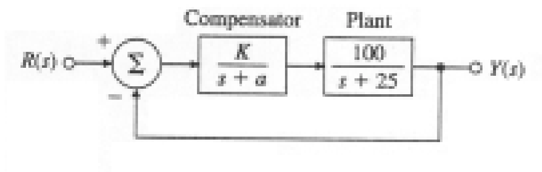


Figure 3.60: Unity feedback system for Problem 3.25

$$\zeta = \sqrt{\frac{(\ln M_p)^2}{\pi^2 + (\ln M_p)^2}} \geq 0.591$$

Using (1) and the solution for ζ:

$$K = \frac{1}{\zeta^2} \leq 2.86$$
$$\therefore 0 < K \leq 2.86$$

25. For the unity feedback system shown in Fig. 3.60, specify the gain and pole location of the compensator so that the overall closed-loop response to a unit-step input has an overshoot of no more than 25%, and a 1% settling time of no more than 0.1 sec. Verify your design using MATLAB.

Solution:

$$\frac{Y(s)}{R(s)} = \frac{100K}{s^2 + (25+a)s + 25a + 100K} = \frac{100K}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Using the given information:

$$\begin{aligned} R(s) &= \frac{1}{s} \qquad \text{unit step} \\ M_p &\leq 25\% \\ t_{1\%} &\leq 0.1\,\text{sec} \end{aligned}$$

Solve for ζ :

$$\begin{aligned} M_p &= e^{-\pi\zeta/\sqrt{1-\zeta^2}} \\ \zeta &= \sqrt{\frac{(\ln M_p)^2}{\pi^2 + (\ln M_p)^2}} \geq 0.4037 \end{aligned}$$

Solve for $\omega_{n:}$

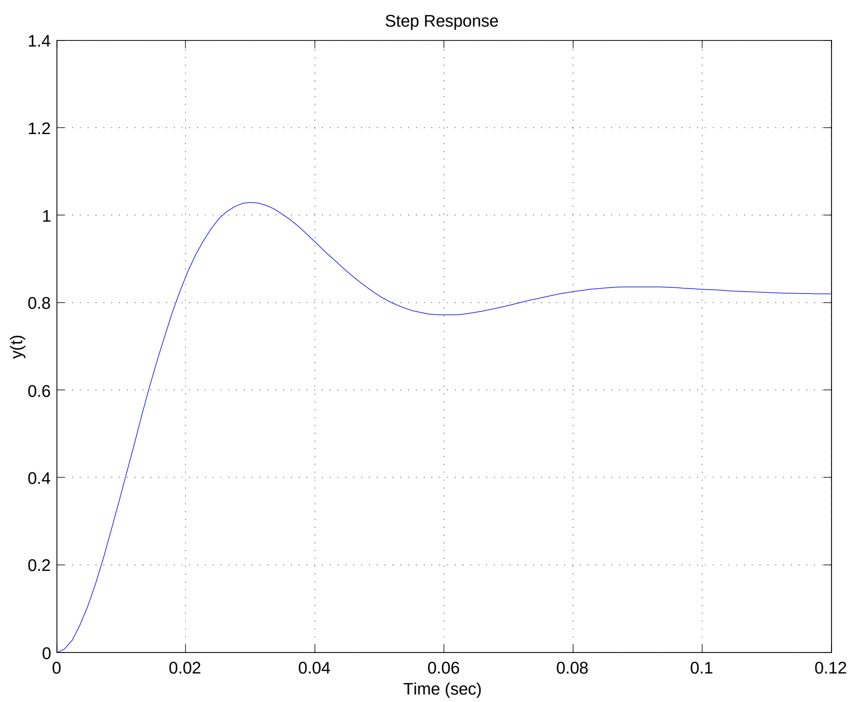
$$e^{-\zeta\omega_n t_s} = 0.01 \qquad \text{For a 1\% settling time}$$

$$t_s \leq \frac{4.605}{\zeta\omega_n} = 0.1$$
$$\Longrightarrow \omega_n \approx 114.07$$

Now find a and K :

$$2\zeta\omega_n = (25 + a)$$
$$a = 2\zeta\omega_n - 25 = 92.10$$
$$\omega_n^2 = (25a + 100K)$$
$$K = \frac{\omega_n^2 - 25a}{100} \approx 107.09$$

The step response of the system using MATLAB is shown below.

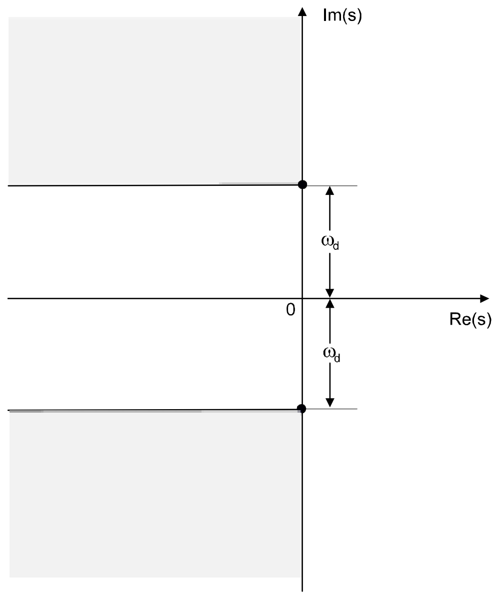


Step Response for Problem 3.25

Problems and Solutions for Section 3.4

26. Suppose you desire the peak time of a given second-order system to be less than t_p'. Draw the region in the s-plane that corresponds to values of the poles that meet the specification $t_p < t_p'$.

Solution:



s-plane region to meet peak time constraint: shaded

$$\omega_d t_p = \pi \Longrightarrow t_p = \frac{\pi}{\omega_d} < t_p^{'}$$
$$\frac{\pi}{t_p'} < \omega_d$$

27. Suppose you are to design a unity feedback controller for a first-order plant depicted in Fig. 3.61. (As you will learn in Chapter 4, the configuration shown is referred to as a proportional-integral controller). You are to design the controller so that the closed-loop poles lie within the shaded regions shown in Fig. 3.62.

(a) What values of ω_n and ζ correspond to the shaded regions in Fig. 3.62? (A simple estimate from the figure is sufficient.)

(b) Let $K_\alpha = \alpha = 2$. Find values for K and K_1 so that the poles of the closed-loop system lie within the shaded regions.

(c) Prove that no matter what the values of K_α and α are, the controller provides enough flexibility to place the poles anywhere in the complex (left-half) plane.

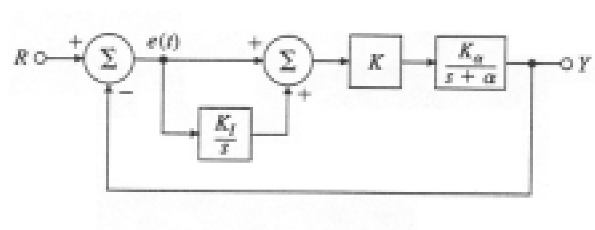


Figure 3.61: Unity feedback system for Problem 3.27

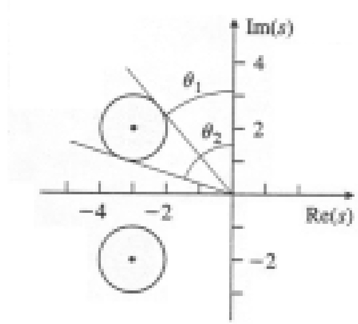


Figure 3.62: Desired closed-loop pole locations for Problem 3.27

Solution:

(a) The values could be worked out mathematically but working from the diagram:

$$\begin{aligned}\sqrt{3^2+2^2} &= 3.6 \Longrightarrow 2.6 \leq \omega_n \leq 4.6\\ \theta &= \sin^{-1}\zeta\\ \zeta &= \sin\theta\end{aligned}$$

From the figure:

$$\begin{aligned}\theta &\approx 34^\circ \qquad \zeta_1 = 0.554\\ \theta &\approx 70^\circ \qquad \zeta_2 = 0.939\end{aligned}$$

$$\Longrightarrow 0.6 \leq \zeta \leq 0.9 \qquad (roughly)$$

(b) Closed-loop pole positions:

$$\begin{aligned}s(s+\alpha)+(Ks+KK_I)K_\alpha &= 0\\ s^2+(\alpha+KK_\alpha)s+KK_IK_\alpha &= 0\end{aligned}$$

For this case:

$$s^2+(2+2K)s+2KK_I = 0 \qquad (*)$$

Choose roots that lie in the center of the shaded region,

$$\begin{gathered}(s+(3+j2))(s+(3-j2)) = s^2+6s+13 = 0\\ s^2+(2+2K)s+2KK_I = s^2+6s+13\\ 2+2K = 6 \Longrightarrow K = 2\\ 13 = 4K_I \Longrightarrow K_I = \frac{13}{4}\end{gathered}$$

(c) For the closed-loop pole positions found in part (b), in the (*) equation the value of K can be chosen to make the coefficient of s take on any value. For this value of K a value of K_I can be chosen so that the quantity KK_IK_α takes on any value desired. This implies that the poles can be placed anywhere in the complex plane.

28. The open-loop transfer function of a unity feedback system is

$$G(s) = \frac{K}{s(s+2)}.$$

The desired system response to a step input is specified as peak time $t_p = 1$ sec and overshoot $M_p = 5\%$.

(a) Determine whether both specifications can be met simultaneously by selecting the right value of K.

(b) Sketch the associated region in the s-plane where both specifications are met, and indicate what root locations are possible for some likely values of K.

(c) Pick a suitable value for K, and use MATLAB to verify that the specifications are satisfied.

Solution:

(a)

$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{K}{s^2+2s+K} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Equate the coefficients:

$$\begin{array}{c} 2 = 2\zeta\omega_n \\ K = \omega_n^2 \end{array} \quad (*)$$

$$\Longrightarrow \quad \omega_n = \sqrt{K} \qquad \zeta = \frac{1}{\sqrt{K}}$$

We would need:

$$\begin{array}{rcl} \frac{M_p\%}{100} = 0.05 = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} & \Longrightarrow & \zeta = 0.69 \\ t_p = 1\,\text{sec} = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} & \Longrightarrow & \omega_n = 4.34 \end{array}$$

But the combination ($\zeta = 0.69$, $\omega_n = 4.34$) that we need is <u>not</u> possible by varying K alone. Observe that from equations (*) $\zeta\omega_n = 1 \neq 0.69 \times 4.34$

(b) Now we wish to have:

$$\begin{array}{c} M_p^* = r \times 0.05 = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \\ t_p^* = r \times 1\,\text{sec} = \frac{\pi}{\omega_d} \end{array} \quad (**)$$

where $r \equiv$relaxation factor.
Recall the conditions of our system:

$$\begin{array}{rcl} \omega_n & = & \sqrt{K} \\ \zeta & = & \frac{1}{\sqrt{K}} \end{array}$$

replace ω_n and ζ in the system (**):

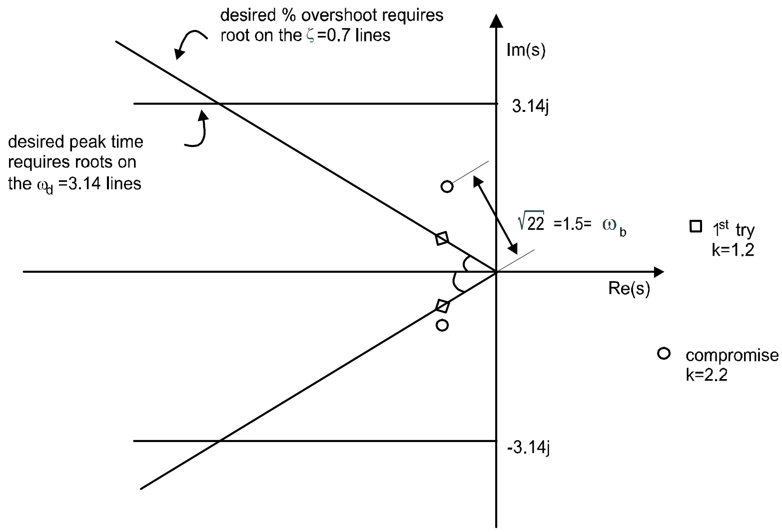
$$\Longrightarrow \quad \begin{array}{c} -\frac{\pi}{\sqrt{K-1}} = r \times 0.05 \\ 1\,\text{sec} = \frac{\pi}{\sqrt{K-1}} \end{array}$$

$$\Longrightarrow r \times 0.05 = e^{-r} \qquad \Longrightarrow \quad r \cong 2.21$$
$$K = 1 + \frac{\pi^2}{r^2} \qquad \Longrightarrow \quad K = 3.02$$

then with $K = 3.02$ we will have:

$$\begin{aligned} M_p^* &= rM_p = 2.21 \times 0.05 = 0.11 \\ t_p^* &= rt_p = 2.21 \times 1\,\text{sec} = 2.21\,\text{sec} \end{aligned}$$

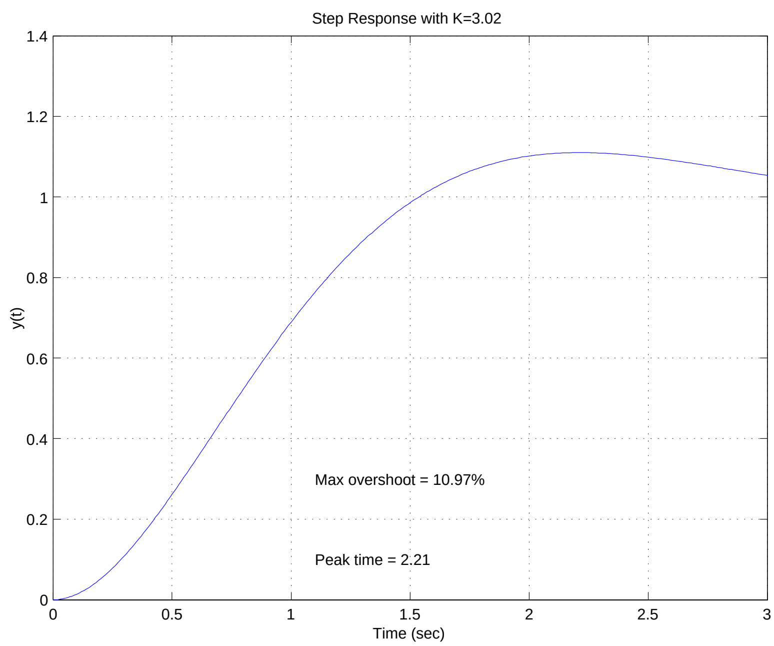
Note: * denotes actual location of closed-loop roots.



s-plane regions

```
% Problem 3.28
K=3.02;
num=[K];
den=[1, 2, K];
sys=tf(num,den);
t=0:.01:3;
y=step(sys,t);
plot(t,y);
yss = dcgain(sys);
Mp = (max(y) - yss)*100;
% Finding maximum overshoot
msg_overshoot = sprintf('Max overshoot = %3.2f%%', Mp);
% Finding peak time
idx = max(find(y==(max(y))));
tp = t(idx);
```

```
msg_peaktime = sprintf('Peak time = %3.2f', tp);
xlabel('Time (sec)');
ylabel('y(t)');
msg_title = sprintf('Step Response with K=%3.2f',K);
title(msg_title);
text(1.1, 0.3, msg_overshoot);
text(1.1, 0.1, msg_peaktime);
grid on;
```



Problem 3.28: Closed-loop step response

29. The equations of motion for the DC motor shown in Fig. 2.26 were given in Eqs. (2.63-64) as

$$J_m\ddot{\theta}_m + \left(b + \frac{K_t K_e}{R_a}\right)\dot{\theta}_m = \frac{K_t}{R_a}v_a.$$

Assume that

$$\begin{aligned}
J_m &= 0.01 \text{ kg} \cdot \text{m}^2, \\
b &= 0.001 \text{ N} \cdot \text{m} \cdot \text{sec}, \\
K_e &= 0.02 \text{ V} \cdot \text{sec}, \\
K_t &= 0.02 \text{ N} \cdot \text{m/A}, \\
R_a &= 10\ \Omega.
\end{aligned}$$

(a) Find the transfer function between the applied voltage v_a and the motor speed $\dot{\theta}_m$.

(b) What is the steady-state speed of the motor after a voltage $v_a = 10$ V has been applied?

(c) Find the transfer function between the applied voltage v_a and the shaft angle θ_m.

(d) Suppose feedback is added to the system in part (c) so that it becomes a position servo device such that the applied voltage is given by

$$v_a = K(\theta_r - \theta_m),$$

where K is the feedback gain. Find the transfer function between θ_r and θ_m.

(e) What is the maximum value of K that can be used if an overshoot $M_p < 20\%$ is desired?

(f) What values of K will provide a rise time of less than 4 sec? (Ignore the M_p constraint.)

(g) Use MATLAB to plot the step response of the position servo system for values of the gain $K = 0.5$, 1, and find the overshoot and rise time of the three step responses by examining your plots. Are the plots consistent with your calculations in parts (e) and (f)?

Solution:

$$J_m\ddot{\theta}_m + \left(b + \frac{K_tK_e}{R_a}\right)\dot{\theta}_m = \frac{K_t}{R_a}v_a$$

(a)

$$J_m\Theta_m s^2 + \left(b + \frac{K_tK_e}{R_a}\right)\Theta_m s = \frac{K_t}{R_a}V_a(s)$$

$$\frac{s\Theta_m(s)}{V_a(s)} = \frac{\frac{K_t}{R_aJ_m}}{s + \frac{b}{J_m} + \frac{K_tK_e}{R_aJ_m}}$$

$$\begin{aligned}
J_m &= 0.01kg \cdot m^2, \\
b &= 0.001N \cdot m \cdot \text{sec}, \\
K_e &= 0.02V \cdot \text{sec}, \\
K_t &= 0.02N \cdot m/A, \\
R_a &= 10\Omega.
\end{aligned}$$

$$\frac{s\Theta_m(s)}{V_a(s)} = \frac{0.2}{s + 0.104}$$

(b) Final Value Theorem

$$\dot{\theta}(\infty) = \frac{s(10)(0.2)}{s(s+0.104)}|_{s=0} = \frac{2}{0.104} = 19.23$$

(c)

$$\frac{\Theta_m(s)}{V_a(s)} = \frac{0.2}{s(s+0.104)}$$

(d)

$$\begin{aligned}
\Theta_m(s) &= \frac{0.2K(\Theta_r - \Theta_m)}{s(s+0.104)} \\
\frac{\Theta_m(s)}{\Theta_r(s)} &= \frac{0.2K}{s^2 + 0.104s + 0.2K}
\end{aligned}$$

(e)

$$\begin{aligned}
M_p &= e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 0.2 \qquad (20\%) \\
\zeta &= 0.4559 \\
Y(s) &= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\
2\zeta\omega_n &= 0.104 \\
\omega_n &= \frac{0.104}{2(0.4559)} = 0.114 rad/\sec \\
\omega_n^2 &= 0.2K \\
K &< 6.50 \times 10^{-2}
\end{aligned}$$

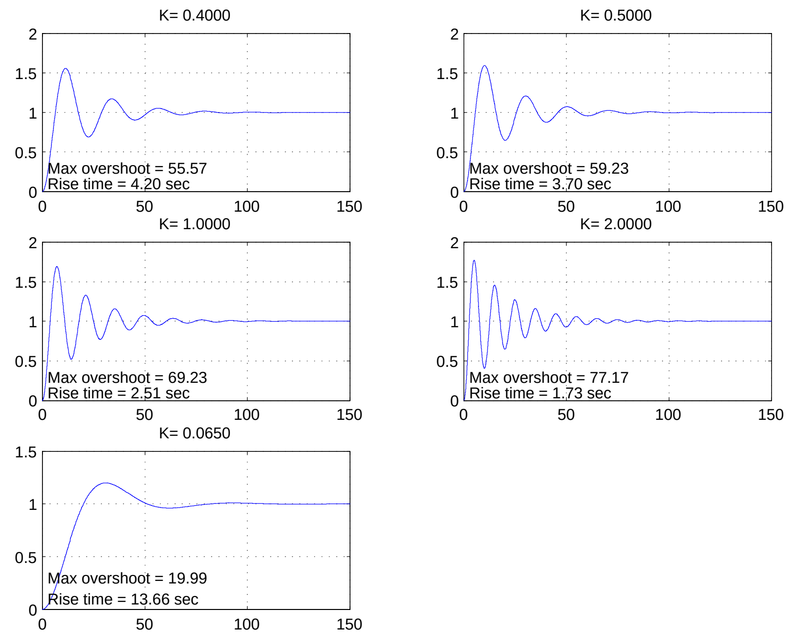
(f)

$$\begin{aligned}
\omega_n &\geq \frac{1.8}{t_r} \\
\omega_n^2 &= 0.2K \\
K &\geq 1.01
\end{aligned}$$

(g) MATLAB

```
clear all
close all
K1=[0.5 1.0 2.0 6.5e-2];
t=0:0.01:150;
for i=1:1:length(K1)
K = K1(i);
titleText = sprintf(' K= %1.4f ', K);
wn = sqrt(0.2*K);
```

```
num=wn^2;

den=[1 0.104 wn^2];

zeta=0.104/2/wn;

sys = tf(num, den);

y= step(sys, t);

% Finding maximum overshoot

if zeta < 1

Mp = (max(y) - 1)*100;

overshootText = sprintf(' Max overshoot = %3.2f %', Mp);

else

overshootText = sprintf(' No overshoot');

end

% Finding rise time

idx_01 = max(find(y<0.1));

idx_09 = min(find(y>0.9));

t_r = t(idx_09) - t(idx_01);

risetimeText = sprintf(' Rise time = %3.2f sec', t_r);

% Plotting

subplot(3,2,i);

plot(t,y);

grid on;

title(titleText);

text( 0.5, 0.3, overshootText);

text( 0.5, 0.1, risetimeText);

end
```



Problem 3.29: Closed-loop step responses

For part (e) we concluded that $K < 6.50 \times 10^{-2}$ in order for $M_p <$ 20%. This is consistent with the above plots. For part (f) we found that $K \geq 1.01$ in order to have a rise time of less than 4 seconds. We actually see that our calculations is slightly off and that K can be $K \geq 0.5$, but since $K \geq 1.01$ is included in $K \geq 0.5$, our answer in part f is consistent with the above plots.

30. You wish to control the elevation of the satellite-tracking antenna shown in Figs. 3.63 and 3.64. The antenna and drive parts have a moment of inertia J and a damping B; these arise to some extent from bearing and aerodynamic friction but mostly from the back emf of the DC drive motor. The equations of motion are

$$J\ddot{\theta} + B\dot{\theta} = T_c,$$

where T_c is the torque from the drive motor. Assume that

$$J = 600,000 \text{kg}\cdot\text{m}^2 \quad B = 20,000 \text{ N}\cdot\text{m}\cdot\text{sec}$$

(a) Find the transfer function between the applied torque T_c and the antenna angle θ.

(b) Suppose the applied torque is computed so that θ tracks a reference command θ_r according to the feedback law

$$T_c = K(\theta_r - \theta),$$

Figure 3.63: Satellite Antenna (Courtesy Space Systems/Loral)

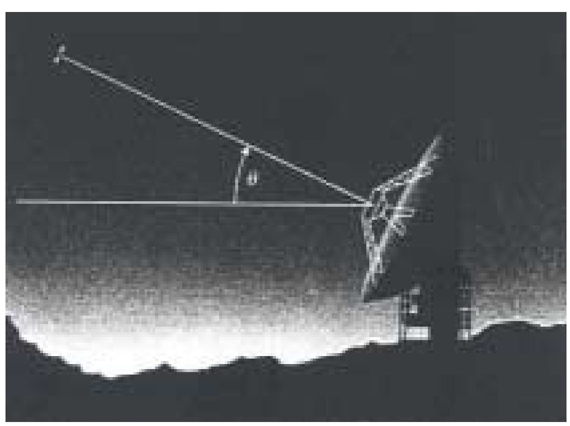


Figure 3.64: Schematic of antenna for Problem 3.30

where K is the feedback gain. Find the transfer function between θ_r and θ.

(c) What is the maximum value of K that can be used if you wish to have an overshoot $M_p < 10\%$?

(d) What values of K will provide a rise time of less than 80 sec? (Ignore the M_p constraint.)

(e) Use MATLAB to plot the step response of the antenna system for $K = 200$, 400, 1000, and 2000. Find the overshoot and rise time of the four step responses by examining your plots. Do the plots confirm your calculations in parts (c) and (d)?

Solution:

$$J\ddot{\theta} + B\dot{\theta} = T_c$$

(a)

$$\begin{aligned}
J\Theta s^2 + B\Theta s &= T_c(s) \\
\frac{\Theta(s)}{T_c(s)} &= \frac{1}{Js+B} \\
J &= 600,000 kg \cdot m^2 \\
B &= 20,000 N \cdot m \cdot \sec \\
\frac{\Theta(s)}{T_c(s)} &= \frac{1.667 \times 10^{-6}}{s(s+\frac{1}{30})}
\end{aligned}$$

(b)

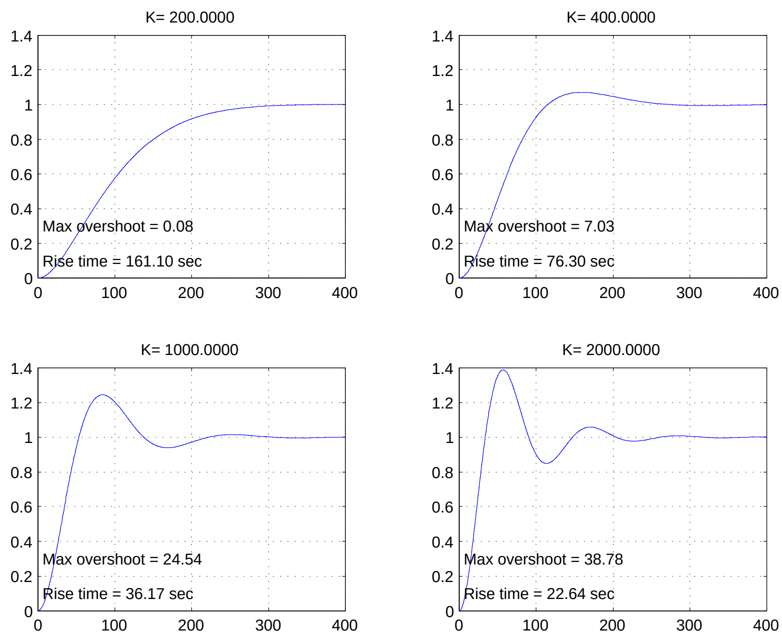
$$\begin{aligned}
\Theta(s) &= \frac{1.667 \times 10^{-6} K(\Theta_r - \Theta)}{s(s+\frac{1}{30})} \\
\frac{\Theta(s)}{\Theta_r(s)} &= \frac{1.667K \times 10^{-6}}{s^2 + \frac{1}{30}s + 1.667K \times 10^{-6}}
\end{aligned}$$

(c)

$$\begin{aligned}
M_p &= e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 0.1 \qquad (10\%) \\
\zeta &= 0.591 \\
Y(s) &= \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\
2\zeta\omega_n &= \frac{1}{30} \\
\omega_n &= \frac{\frac{1}{30}}{2(0.591)} = 0.0282\ rads/\sec \\
\omega_n^2 &= 1.667K \times 10^{-6} \\
K &< 477
\end{aligned}$$

(d)

$$\begin{aligned} \omega_n &\geq \frac{1.8}{t_r} \\ \omega_n^2 &= 1.667K \times 10^{-6} \\ K &\geq 304 \end{aligned}$$



(e) Problem 3.30: Step responses

(e) The results compare favorably with the predictions made in parts c and d. For $K < 477$ the overshoot was less than 10, the rise-time was less than 80 seconds.

31. (a) Show that the second-order system

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = 0, \quad y(0) = y_o, \quad \dot{y}(0) = 0,$$

has the response

$$y(t) = y_o \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \cos^{-1}\zeta).$$

(b) Prove that, for the underdamped case ($\zeta < 1$), the response oscillations decay at a predictable rate (see Fig. 3.65) called the **logarithmic decrement** δ, where

$$\begin{aligned} \delta &= \ln \frac{y_o}{y_1} = \sigma\tau_d \\ &= \ln \frac{\Delta y_1}{y_1} \cong \ln \frac{\Delta y_i}{y_i}, \end{aligned}$$

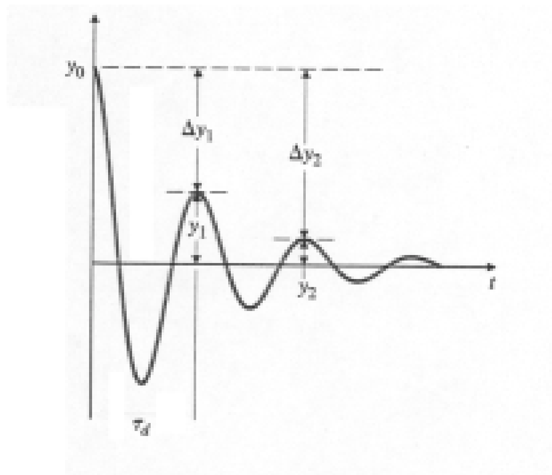


Figure 3.65: Definition of logarithmic decrement

and τ_d is the damped natural period of vibration

$$\tau_d = \frac{2\pi}{\omega_d}.$$

Solution:

(a) The system is second order $\Longrightarrow Q(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$. The initial condition response can be obtained by plugging a dirac delta at the input at the time 0 (this "charges" the system immediately to its initial condition and after that the system evolves by itself).

$$\begin{aligned} \text{Input}_{effective} &= y_0\delta(t) \\ \mathcal{L}[\text{Input}_{effective}] &= y_0 \end{aligned}$$

We do not know whether the transfer function has finite zeros or not, but further thought will reveal the presence of at least one finite zero in the $H(s)$.

$$\lim_{s\to\infty} sH(s)y_0 = y(t)|_{0+}$$

where

$$H(s) = \frac{P(s)}{Q(s)} = \frac{P(s)}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

If $P(s)$ were a constant (no zeros in the $H(s)$), then the limit in the initial value theorem would give always zero (which is wrong because we know that the initial value must be y_0.) So we need a zero. We suggest using the following $H(s)$:

$$\begin{aligned} H(s) &= \frac{-s}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ Y(s) &= H(s)y_0 = \frac{-sy_0}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ &= \frac{R_+}{s - P_+} + \frac{R_-}{s - P_-} \end{aligned}$$

where

$$\begin{aligned} P_+ &= -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2} \\ P_- &= -\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2} \\ R_+ &= \frac{-\omega_n e^{j(\pi = \cos^{-1}\zeta)}}{2\omega_n\sqrt{1-\zeta^2}e^{j\pi/s}} \\ R_- &= R_+^* \end{aligned}$$

Note: The residues can be calculated graphically.

$$R_+ = \lim_{s \to P_+} [(s - P_+)Y(s)]$$

$$\Longrightarrow \qquad y(t) = R_+ e^{P_+ t} + R_- e^{P_- t}$$

$$y(t) = \frac{-e^{-\zeta\omega_n t}}{2\sqrt{1-\zeta^2}}[e^{+j(\omega_n\sqrt{1-\zeta^2}t + \pi/2 - \cos^{-1}\zeta)} + e^{-j(\omega_n\sqrt{1-\zeta^2}t + \pi/2 - \cos^{-1}\zeta)}]$$

$$\Longrightarrow y(t) = y_0 \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t - \cos^{-1}\zeta)$$

(b)

$$\begin{aligned} \frac{dy(t)}{dt} &= 0 \qquad \Longrightarrow t = \frac{n\pi}{\omega_d} \qquad \text{(n is any integer)} \\ t_{Max} &= \frac{2\pi}{\omega_d}n \\ y(t)|_{t_{Max}} &\equiv y_n = y_0 \frac{e^{-\sigma n \tau_d}}{\sqrt{1-\zeta^2}} \sin(\cos^{-1}\zeta) \end{aligned}$$

Note:

$$\sin(-\cos^{-1}\zeta) = \sqrt{1-\zeta^2}$$

$$y_n = \frac{y_0\sqrt{1-\zeta^2}}{\sqrt{1-\zeta^2}} e^{-\sigma n \tau_d} \qquad (*)$$

(Proof of the first line)

$$\sigma = \ln \frac{y_0}{y_n} = \sigma \tau_d$$

From (*)

$$y_1 = y_0 e^{-\sigma\tau_d} \Longrightarrow \ln \frac{y_0}{y_n} = \sigma\tau_d$$

(Proof of the second line)

$$\begin{aligned} \Delta y_n &= y_{n-1} - y_n \\ \Delta y_n &= y_0 e^{-\sigma n \tau_d} - y_0 e^{-(n-1)\sigma\tau_d} = y_0 e^{-\sigma n \tau_d}(1 - e^{\sigma\tau_d}) \end{aligned}$$

$$\begin{aligned} \Longrightarrow \quad \frac{\Delta y_n}{y_n} &= \frac{y_0 e^{-\sigma n \tau_d}}{y_0 e^{-\sigma n \tau_d}}(1 - e^{\sigma\tau_d}) \\ \Longrightarrow \quad \frac{\Delta y_n}{y_n} &= \frac{\Delta y_i}{yi} \qquad \text{for all } i, n \end{aligned}$$

Problems and Solutions for Section 3.5

32. In aircraft control systems, an ideal pitch response (q_o) versus a pitch command (q_c) is described by the transfer function

$$\frac{Q_o(s)}{Q_c(s)} = \frac{\tau\omega_n^2(s + 1/\tau)}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

The actual aircraft response is more complicated than this ideal transfer function; nevertheless, the ideal model is used as a guide for autopilot design. Assume that t_r is the desired rise time, and that

$$\begin{aligned} \omega_n &= \frac{1.789}{t_r} \\ \frac{1}{\tau} &= \frac{1.6}{t_r} \\ \zeta &= 0.89 \end{aligned}$$

Show that this ideal response possesses a fast settling time and minimal overshoot by plotting the step response for $t_r = 0.8$, 1.0, 1.2, and 1.5 sec.

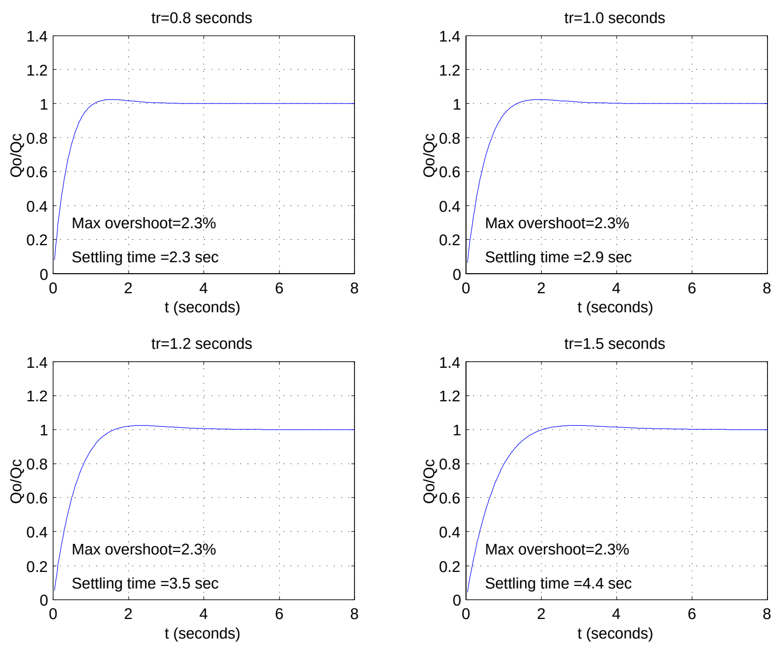
Solution:

The following program statements in MATLAB produce the following plots:

```
% Problem 3.32
```

```
tr = [0.8 1.0 1.2 1.5];
t=[1:240]/30;
tback=fliplr(t);
clf;
for I=1:4,
    wn=(1.789)/tr(I); %Rads/second
    tau=tr(I)/(1.6); %tau
    zeta=0.89; %
    b=tau*(wn^2)*[1 1/tau];
    a=[1 2*zeta*wn (wn^2)];
    y=step(b,a,t);
    subplot(2,2,I);
    plot(t,y);
    titletext=sprintf('tr=%3.1f seconds',tr(I));
    title(titletext);
    xlabel('t (seconds)');
    ylabel('Qo/Qc');
    ymax=(max(y)-1)*100;
    msg=sprintf('Max overshoot=%3.1f%%',ymax);
    text(.50,.30,msg);
    yback=flipud(y);
    yind=find(abs(yback-1)>0.01);
    ts=tback(min(yind));
     msg=sprintf('Settling time =%3.1f sec',ts);
    text(.50,.10,msg);
     grid;
end
```

Figure 3.66: Fig.3.66: Unity feedback system for Problem 3.33



Problem 3.32: Ideal pitch response

33. Consider the system shown in Fig. 3.66, where

$$G(s) = \frac{1}{s(s+3)} \quad \text{and} \quad D(s) = \frac{K(s+z)}{s+p}. \tag{1}$$

Find K, z, and p so that the closed-loop system has a 10% overshoot to a step input and a settling time of 1.5 sec (1% criterion).

Solution:

For the 10% overshoot:

$$\begin{aligned} M_p &= e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 10\% \\ \Longrightarrow &\quad \zeta = \sqrt{\frac{(\ln M_p)^2}{\pi^2 + (\ln M_p)^2}} = 0. \end{aligned}$$

For the 1.5sec (1% criterion):

$$\omega_n = \frac{4.6}{\zeta t_s} = \frac{4.6}{(0.6)(1.5)} = 5.11$$

The closed loop transfer function is:

$$\frac{Y(s)}{R(s)} = \frac{K\frac{s+z}{s+p} \times \frac{1}{s(s+3)}}{1 + K\frac{s+z}{s+p} \times \frac{1}{s(s+3)}} = \frac{K(s+z)}{s(s+3)(s+p) + K(s+z)}$$

Method I.

From inspection, if $z = 3$, $(s+3)$ will cancel out and we will have a standard form transfer function. As perfect cancellation is impossible, assign z a value that is very close to 3, say 3.1. But in determining the K and p, assume that $(s+3)$ and$(s+3.1)$ cancelled out each other. Then:

$$\frac{Y(s)}{R(s)} = \frac{K}{s^2 + ps + K}$$

As the additional pole and zero will degrade the system, pick some larger damping ration.

Let $\zeta = 0.7$

$$\begin{aligned} \omega_n &= \frac{4.6}{\zeta t_s} = \frac{4.6}{(0.7)(1.5)} = 4.38, \text{ so let } \omega_n = 4.5 \\ p &= 2\zeta\omega_n = 2 \times 0.7 \times 4.5 = 6.3 \\ K &= \omega_n^2 = 20.25 \end{aligned}$$

Method II.

There are 3 unknowns (z, p, K) and only 2 specified conditions. We can arbitrarily choose p large such that complex poles will dominate in the system response.

Try $p = 10z$

Choose a damping ratio corresponding to an overshoot of 5% (instead of 10%, to be safe).

$\zeta = 0.707$

From the formula for settling time (with a 1% criterion)

$\omega_n = \frac{4.6}{\zeta t_s} = \frac{4.6}{0.707 \times 1.5} = 4.34$ adding some margin, let $\omega_n = 4.88$

The characteristic equation is

$$Q(s) = s^3 + (s+p)\, s^2 + (3p+K)s + Kz = (s+a)(s^2 + 2\zeta\omega_n s + \omega_n^2)$$

We want the characteristic equation to be the product of two factors, a couple of conjugated poles (dominant) and a non-dominant real pole far form the dominant poles.

Equate both expressions of the characteristic equation.

$$\begin{aligned}
\omega_n^2 a &= Kz \\
2\zeta\omega_n a + \omega_n^2 &= 30z + K \\
2\zeta\omega_n + a &= 3 + 10z
\end{aligned}$$

Solving three equations we get

$$\begin{aligned}
z &= 5.77 \\
p &= 57.7 \\
K &= 222.45 \\
a &= 53.79
\end{aligned}$$

34. Sketch the step response of a system with the transfer function

$$G(s) = \frac{s/2 + 1}{(s/40 + 1)[(s/4)^2 + s/4 + 1]}.$$

Justify your answer based on the locations of the poles and zeros (do not find inverse Laplace transform). Then compare your answer with the step response computed using MATLAB.

Solution:

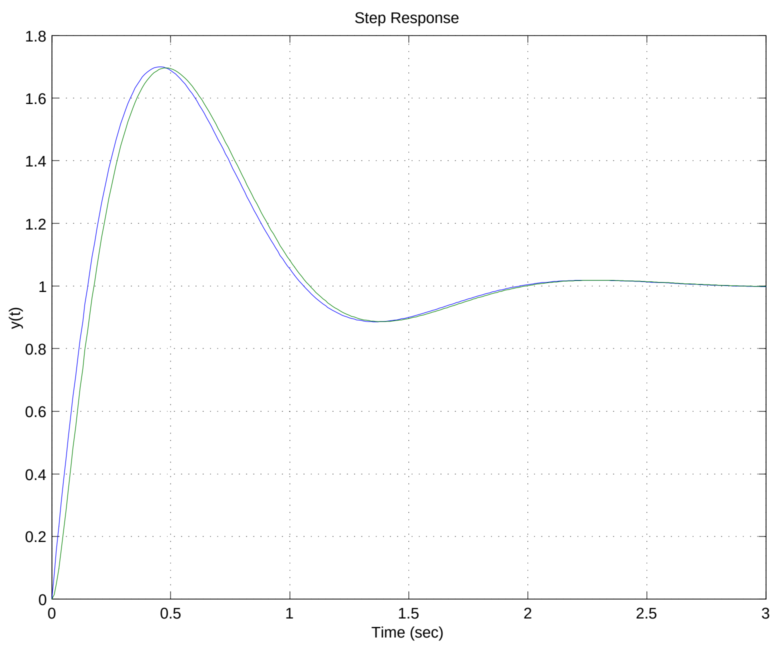
From the location of the poles, we notice that the real pole is a factor of 20 away from the complex pair of poles. Therefore, the reponse of the system is dominated by the complex pair of poles.

$$G(s) \approx \frac{(s/2 + 1)}{[(s/4)^2 + s/4 + 1]}$$

This is now in the same form as equation (3.58) where $\alpha = 1$, $\zeta = 0.5$ and $\omega_n = 4$. Therefore, Fig. 3.32 suggests an overshoot of over 70%. The step response is the same as shown in Fig. 3.31, for $\alpha = 1$, with more than 70% overshoot and settling time of 3 seconds. The MATLAB plots below confirm this.

```
% Problem 3.34
num=[1/2, 1];
den1=[1/16, 1/4, 1];
```

```
sys1=tf(num,den1);
t=0:.01:3;
y1=step(sys1,t);
den=conv([1/40, 1],den1);
sys=tf(num,den);
y=step(sys,t);
plot(t,y1,t,y);
xlabel('Time (sec)');
ylabel('y(t)');
title('Step Response');
grid on;
```



Problem 3.34: Step responses

35. Consider the two nonminimum phase systems,

$$G_1(s) = -\frac{2(s-1)}{(s+1)(s+2)}; \tag{2}$$

$$G_2(s) = \frac{3(s-1)(s-2)}{(s+1)(s+2)(s+3)}. \tag{3}$$

(a) Sketch the unit step responses for $G_1(s)$ and $G_2(s)$, paying close attention to the transient part of the response.

(b) Explain the difference in the behavior of the two responses as it relates to the zero locations.

(c) Consider a stable, strictly proper system (that is, m zeros and n poles, where $m < n$). Let $y(t)$ denote the step response of the system. The step response is said to have an undershoot if it initially starts off in the "wrong" direction. Prove that a stable, strictly proper system has an undershoot if and only if its transfer function has an **odd** number of **real** RHP zeros.

Solution:

(a) For $G_1(s)$:

$$\begin{aligned}
Y_1(s) &= \frac{1}{s}G_1(s) = \frac{-2(s-1)}{s(s+1)(s+2)} \\
H(s) &= k\frac{\prod^j (s-z_j)}{\prod^l (s-p_l)} \\
R_{p_i} &= \lim_{s\to p_i}[(s-p_i)H(s)] = \lim_{s\to p_i} k\frac{\prod^j (s-z_j)}{\prod^l_{l\neq i}(s-p_l)} = k\frac{\prod^j (p_i-z_j)}{\prod^l_{l\neq i}(p_i-p_l)}
\end{aligned}$$

Each factor $(p_i - z_j)$ or $(p_i - p_l)$ can be thought of as a complex number (a magnitude and a phase) whose pictorial representation is a vector pointing to p_i and coming from z_j or p_l respectively.

The method for calculating the residue at a pole p_i is:

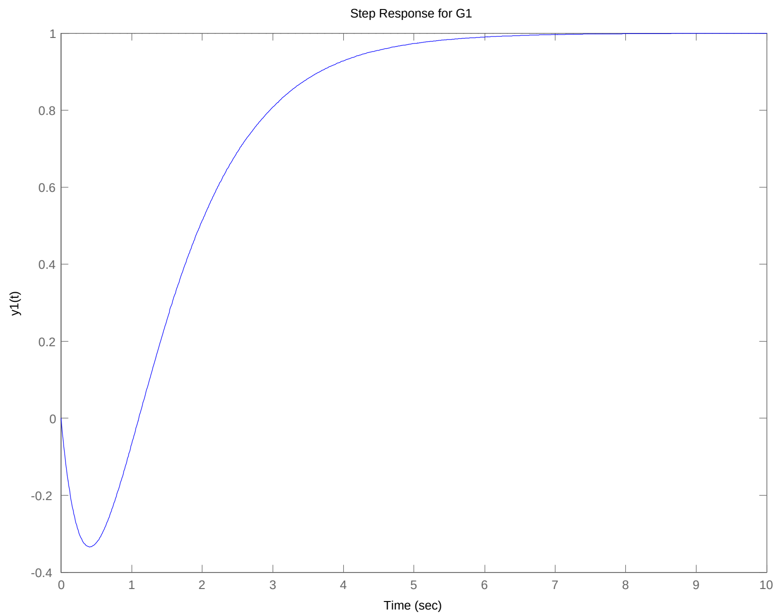
(1) Draw vectors from the rest of the poles and from all the zeros to the pole p_i.

(2) Measure magnitude and phase of these vectors.

(3) The residue will be equal to the gain, multiplied by the product of the vectors coming from the zeros and divided by the product of the vectors coming from the poles.

In our problem:

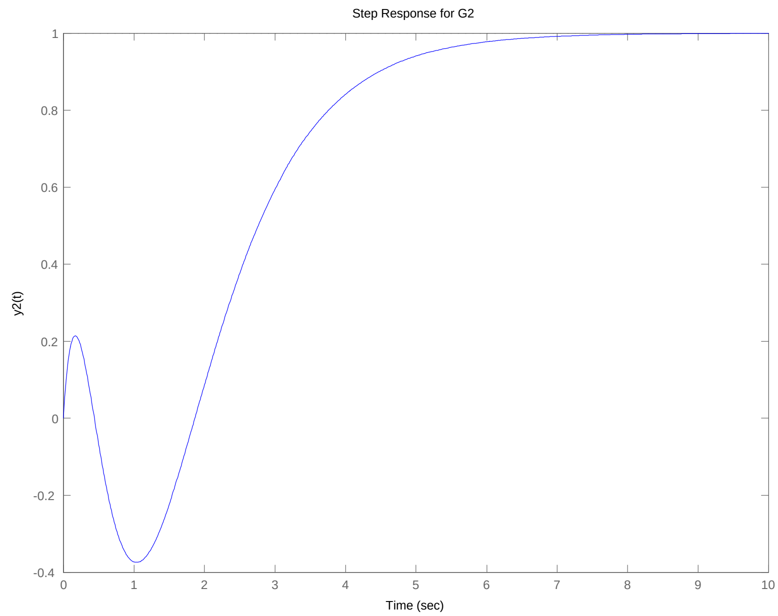
$$\begin{aligned}
Y_1(s) &= \frac{-2(s-1)}{s(s+1)(s+2)} = \frac{R_0}{s} + \frac{R_{-1}}{(s+1)} + \frac{R_{-2}}{(s+2)} = \frac{1}{s} - \frac{4}{s+1} + \frac{3}{s+2} \\
y_1(t) &= 1 - 4e^{-t} + 3e^{-2t}
\end{aligned}$$



Problem 3.35: Step response for a non-minimum phase system

For $G_2(s)$:

$$
\begin{aligned}
Y_2(s) &= \frac{3(s-1)(s-2)}{s(s+1)(s+2)(s+3)} = \frac{1}{s} + \frac{-9}{(s+1)} + \frac{18}{(s+2)} + \frac{-10}{(s+3)} \\
y_2(t) &= 1 - 9e^{-t} + 18e^{-2t} - 10e^{-3t}
\end{aligned}
$$



Problem 3.35: Step response of non-minimum phase system

(b) The first system presents an "undershoot". The second system, on the other hand, starts off in the right direction.

The reasons for this initial behavior of the step response will be analyzed in part c.

In $y_1(t)$: dominant at $t = 0$ the term $-4e^{-t}$

In $y_2(t)$: dominant at $t = 0$ the term $18e^{-2t}$

(c) The following concise proof is from [1] (see also [2]-[3]).

Without loss of generality assume the system has unity DC gain ($G(0) = 1$) Since the system is stable, $y(\infty) = G(0) = 1$, and it is reasonable to assume $y(\infty) \neq 0$. Let us denote the pole-zero excess as $r = n - m$. Then, $y(t)$ and its $r - 1$ derivatives are zero at $t = 0$, and $y^r(0)$ is the first non-zero derivative. The system has an undershoot if $y^r(0)y(\infty) < 0$. The transfer function may be re-written as

$$G(s) = \frac{\prod_{i=1}^{m}(1 - \frac{s}{z_i})}{\prod_{i=1}^{m+r}(1 - \frac{s}{p_i})}$$

The numerator terms can be classified into three types of terms:

(1). The first group of terms are of the form $(1 - \alpha_i s)$ with $\alpha_i > 0$.

(2). The second group of terms are of the form $(1 + \alpha_i s)$ with $\alpha_i > 0$.

(3). Finally, the third group of terms are of the form, $(1 + \beta_i s + \alpha_i s^2)$ with $\alpha_i > 0$, and β_i could be negative.

However, $\beta_i^2 < 4\alpha_i$, so that the corresponding zeros are complex.

All the **denominator** terms are of the form (2), (3), above. Since,

$$y^r(0) = \lim_{s\to\infty} s^r G(s)$$

it is seen that the **sign** of $y^r(0)$ is determined entirely by the number of terms of group 3 above. In particular, if the number is **odd**, then $y^r(0)$ is **negative** and if it is even, then $y^r(0)$ is positive. Since $y(\infty) = G(0) = 1$,then we have the desired result.

[1] Vidyasagar, M., "On Undershoot and Nonminimum Phase Zeros," *IEEE Trans. Automat. Contr.*, Vol. AC-31, p. 440, May 1986.

[2] Clark, R., N., *Introduction to Automatic Control Systems*, John Wiley, 1962.

[3] Mita, T. and H. Yoshida, "Undershooting phenomenon and its control in linear multivariable servomechanisms, " *IEEE Trans. Automat. Contr.*, Vol. AC-26, pp. 402-407, 1981.

36. Consider the following second-order system with an extra pole:

$$H(s) = \frac{\omega_n^2 p}{(s+p)(s^2 + 2\zeta\omega_n s + \omega_n^2)}.$$

Show that the unit step response is

$$y(t) = 1 + Ae^{-pt} + Be^{-\sigma t}\sin(\omega_d t - \theta),$$

where

$$A = \frac{-\omega_n^2}{\omega_n^2 - 2\zeta\omega_n p + p^2}$$
$$B = \frac{p}{\sqrt{(p^2 - 2\zeta\omega_n p + \omega_n^2)(1-\zeta^2)}}$$
$$\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{-\zeta} + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{p - \zeta\omega_n}.$$

(a) Which term dominates $y(t)$ as p gets large?

(b) Give approximate values for A and B for small values of p.

(c) Which term dominates as p gets small? (Small with respect to what?)

(d) Using the explicit expression for $y(t)$ above or the step command in MATLAB, and assuming $\omega_n = 1$ and $\zeta = 0.7$, plot the step response of the system above for several values of p ranging from very small to very large. At what point does the extra pole cease to have much effect on the system response?

Solution:

Second-order system:

$$H(s) = \frac{\omega_n^2 p}{(s+p)(s^2+2\zeta\omega_n s+\omega_n^2)}$$

Unit step response:

$$Y(s) = \frac{1}{s}H(s), \quad y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$s^2+2\zeta\omega_n s+\omega_n^2 = (s+\sigma+j\omega_d)(s+\sigma-j\omega_d)$$

where $\sigma = \zeta\omega_n, \omega_d = \omega_n\sqrt{1-\zeta^2}$.

Thus from partial fraction expansion:

$$Y(s) = \frac{k_1}{s} + \frac{k_2}{s+p} + \frac{k_3}{s+\sigma+j\omega_d} + \frac{k_4}{s+\sigma-j\omega_d}$$

solving for k_1, k_2, k_3, and k_4:

$$\begin{aligned}
k_1 &= H(0) \Longrightarrow k_1 = 1 \\
k_2 &= \frac{\omega_n^2 p}{s(s+\sigma+j\omega_d)(s+\sigma-j\omega_d)}|_{s=-p} \Longrightarrow k_2 = \frac{-\omega_n^2}{\omega_n^2 - 2p\zeta\omega_n + p^2} \\
k_3 &= (s+\sigma+j\omega_d)Y(s)|_{s=-\sigma-j\omega_d} \\
\Longrightarrow & \quad k_3 = \frac{p}{2\sqrt{(1-\zeta^2)(p^2-2p\zeta\omega_n+\omega_n^2)}} e^{-i\theta} = |k_3|\, e^{-i\theta} \\
k_4 &= k_3^*
\end{aligned}$$

where

$$\theta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{-\zeta}\right) + \tan^{-1}\left(\frac{\omega_n\sqrt{1-\zeta^2}}{p-\zeta\omega_n}\right)$$

Thus

$$Y(s) = \frac{1}{s} + \frac{k_2}{s+p} + |k_3|\left(\frac{e^{-i\theta}}{s+\sigma+j\omega_d} + \frac{e^{+i\theta}}{s+\sigma-j\omega_d}\right)$$

Inverse Laplace:

$$y(t) = 1 + k_2 e^{-pt} + |k_3|\left(e^{-i\theta}e^{-(\sigma+j\omega_d)t} + e^{+i\theta}e^{-(\sigma-j\omega_d)t}\right)$$

or

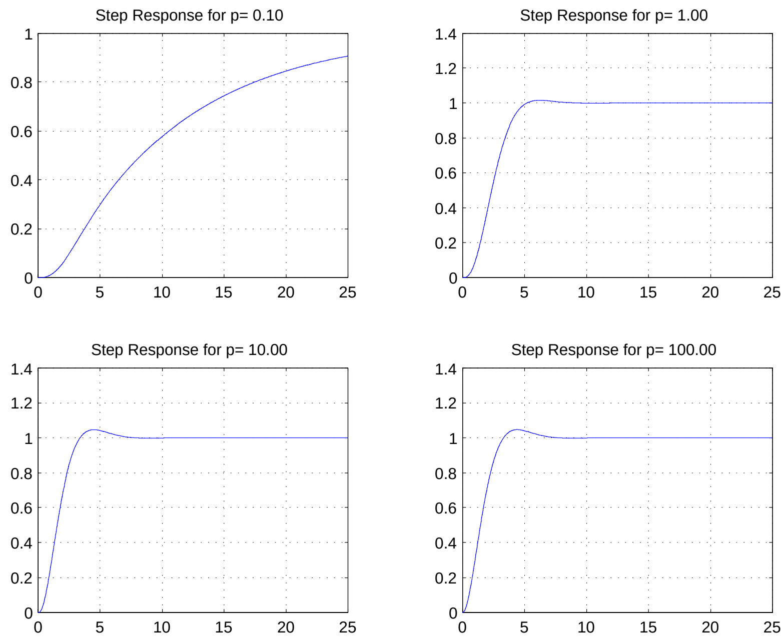
$$y(t) = 1 + \underbrace{\frac{-\omega_n^2}{\omega_n^2 - 2p\zeta\omega_n + p^2}}_{A} e^{-pt} + \underbrace{\frac{p}{\sqrt{(1-\zeta^2)(p^2-2p\zeta\omega_n+\omega_n^2)}}}_{B} e^{-\sigma t}\cos(\omega_d t + \theta)$$

(a) As p gets large the B term dominates.

(b) For small p: $A \approx -1, B \approx 0$.

(c) As p gets small A dominates.

(d) The effect of a change in p is not noticeable above $p \approx 10$.



Problem 3.36: Step responses

37. The block diagram of an autopilot designed to maintain the pitch attitude θ of an aircraft is shown in Fig. 3.67. The transfer function relating the elevator angle δ_e and the pitch attitude θ is

$$\frac{\theta(s)}{\delta_e(s)} = G(s) = \frac{50(s+1)(s+2)}{(s^2+5s+40)(s^2+0.03s+0.06)},$$

where θ is the pitch attitude in degrees and δ_e is the elevator angle in degrees. The autopilot controller uses the pitch attitude error ε to adjust the elevator according to the following transfer function:

$$\frac{\delta_e(s)}{\varepsilon(s)} = D(s) = \frac{K(s+3)}{s+10}.$$

Using MATLAB, find a value of K that will provide an overshoot of less than 10% and a rise time faster than 0.5 sec for a unit step change in θ_r. After examining the step response of the system for various values of K, comment on the difficulty associated with making rise-time and overshoot

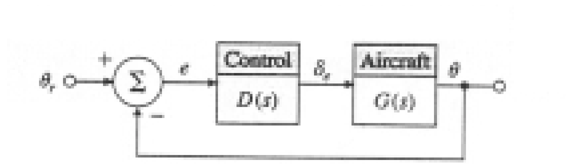


Figure 3.67: Block diagram of autopilot

measurements for complicated systems.

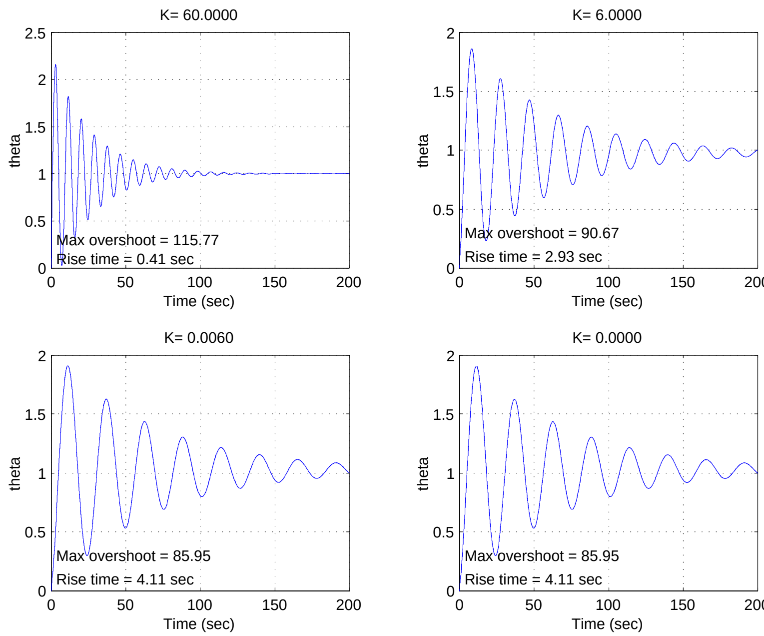
Solution:

$$
\begin{aligned}
G(s) &= \frac{\Theta(s)}{\delta_e(s)} = \frac{50\,(s+1)\,(s+2)}{(s^2+5s+40)\,(s^2+0.03s+0.06)} \\
D(s) &= \frac{\delta_e(s)}{e(s)} = \frac{K(s+3)}{(s+10)}
\end{aligned}
$$

where

$$
\begin{aligned}
e(s) &= \Theta_r - \Theta \\
\frac{\Theta}{\Theta_r} &= \frac{G(s)D(s)}{1+G(s)D(s)} \\
&= \frac{50K\,(s+1)\,(s+2)\,(s+3)}{(s^2+5s+40)\,(s^2+0.03s+0.06)\,(s+10)+K\,(s+3)} \\
&= \frac{50K\left(s^3+6s^2+11s+6\right)}{s^5+15.03s^4+90.51s^3+403.6s^2+(17.4+K)\,s+(24+3K)}
\end{aligned}
$$

Output must be normalized to the final value of $\frac{\Theta(s)}{\Theta_r(s)}$ for easy computation of the overshoot and rise-time. In this case the design criterion for overshoot cannot be met which is indicated in the sample plots.



Problem 3.37: Step responses for autopilot

Problems and Solutions for Section 3.6

38. A measure of the degree of instability in an unstable aircraft response is the amount of time it takes for the amplitude of the time response to double (see Fig. 3.68) given some nonzero initial condition.

   (a) For a first-order system, show that the time to double τ_2 is

   $$\tau_2 = \frac{\ln 2}{p},$$

   where p is the pole location in the RHP.

   (b) For a second-order system (with two complex poles in the RHP), show that

   $$\tau_2 = \frac{\ln 2}{-\zeta\omega_n}.$$

Solution:

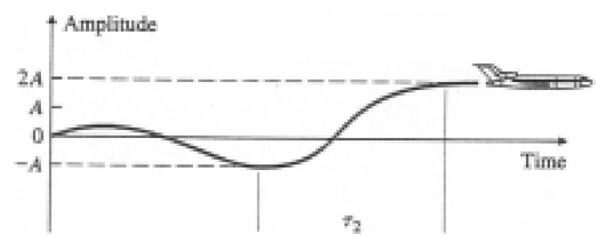


Figure 3.68: Time to double

(a) First-order system, $H(s)$ could be:

$$\begin{aligned} H(s) &= \frac{k}{(s-p)} \\ h(t) &= \mathcal{L}^{-1}\left[H(s)\right] = ke^{pt} \\ h(\tau_0) &= ke^{p\tau_0} \\ h\left(\tau_0+\tau_2\right) &= 2h\left(\tau_0\right) = ke^{p(\tau_0+\tau_2)} \end{aligned}$$

$$\Longrightarrow \qquad 2ke^{p\tau_0} = ke^{p\tau_0}e^{p\tau_2}$$
$$\Longrightarrow \qquad \tau_2 = \frac{\ln 2}{p}$$

(b) Second-order system:

$$y\left(t\right) = y_0 \frac{e^{\omega_n|\zeta|t}}{\sqrt{1-\left|\zeta^2\right|}} \sin(\omega_n\sqrt{1-\left|\zeta^2\right|}t + \cos^{-1}\zeta)$$

where

$$\cos^{-1}\zeta = \cos^{-1}|\zeta| + \pi$$

$$\Longrightarrow y\left(t\right) = y_0 \frac{e^{\omega_n|\zeta|t}}{\sqrt{1-\left|\zeta^2\right|}} (-1)\sin\left(\omega_n\sqrt{1-\left|\zeta^2\right|}t + \cos^{-1}|\zeta|\right)$$

Note: Instead of working with a negative ζ, everything is changed to $|\zeta|$.

$$
\begin{aligned}
|t_0| &= -y_0 \frac{e^{\omega_n |\zeta| t}}{\sqrt{1-|\zeta^2|}} \\
|\tau_0| &= -y_0 \frac{e^{\omega_n |\zeta| \tau_0}}{\sqrt{1-|\zeta^2|}} \\
|\tau_0+\tau_2| &= -y_0 \frac{e^{\omega_n |\zeta| (\tau_0+\tau_2)}}{\sqrt{1-|\zeta^2|}} = 2\,|\tau_0|
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow \qquad & e^{\omega_n |\zeta| \tau_2} = 2 \\
\Longrightarrow \qquad & \tau_2 = \frac{\ln 2}{\omega_n |\zeta|} = \frac{\ln 2}{-\omega_n \zeta} \qquad (\zeta \le 0)
\end{aligned}
$$

Note: This problem shows that $\sigma = \omega_n |\zeta|$ (the real part of the poles) is inversely proportional to the time to double.

The further away from the imaginary axis the poles lie, the faster the response is (either increasing faster for RHP poles or decreasing faster for LHP poles).

39. Suppose that unity feedback is to be applied around the following open-loop systems. Use Routh's stability criterion to determine whether the resulting closed-loop systems will be stable.

(a) $KG(s) = \frac{4(s+2)}{s(s^3+2s^2+3s+4)}$

(b) $KG(s) = \frac{2(s+4)}{s^2(s+1)}$

(c) $KG(s) = \frac{4(s^3+2s^2+s+1)}{s^2(s^3+2s^2-s-1)}$

Solution:

(a)

$$1 + KG = s^4 + 2s^3 + 3s^2 + 8s + 8 = 0$$

$$
\begin{array}{lclll}
s^4 & : & 1 & 3 & 8 \\
s^3 & : & 2 & 8 & \\
s^2 & : & a & b & \\
s^1 & : & c & & \\
s^0 & : & d & &
\end{array}
$$

where

$$
\begin{aligned}
a &= \frac{2 \times 3 - 8 \times 1}{2} = -1 \qquad b = \frac{2 \times 8 - 1 \times 0}{2} = 8 \\
c &= \frac{3a - 2b}{a} = \frac{-8 - 16}{-1} = 24 \\
d &= b = 8
\end{aligned}
$$

2 sign changes in first column$\Longrightarrow$2 roots not in LHP$\Longrightarrow$unstable.

(b)

$$1 + KG = s^3 + s^2 + 2s + 8 = 0$$

The Routh's array is,

$$
\begin{array}{llll}
s^3 & : & 1 & 2 \\
s^2 & : & 1 & 8 \\
s^1 & : & -6 & \\
s^0 & : & 8 &
\end{array}
$$

There are two sign changes in the first column of the Routh array. Therefore, there are two roots not in the LHP.

(c)

$$1 + KG = s^5 + 2s^4 + 3s^3 + 7s^2 + 4s + 4 = 0$$

$$
\begin{array}{lllll}
s^5 & : & 1 & 3 & 4 \\
s^4 & : & 2 & 7 & 4 \\
s^3 & : & a_1 & a_2 & \\
s^2 & : & b_1 & b_2 & \\
s^1 & : & c_1 & & \\
s^0 & : & d_1 & &
\end{array}
$$

where

$$
\begin{aligned}
a_1 &= \frac{6 - 7}{2} = \frac{-1}{2} \qquad a_2 = \frac{8 - 4}{2} = 2 \\
b_1 &= \frac{-7/2 - 4}{-1/2} = 15 \qquad b_2 = \frac{-4/2 - 0}{-1/2} = 4 \\
c_1 &= \frac{30 + 2}{15} = \frac{32}{15} \\
d_1 &= 4
\end{aligned}
$$

2 sign changes in the first column$\Longrightarrow$2 roots not in the LHP$\Longrightarrow$unstable.

40. Use Routh's stability criterion to determine how many roots with positive real parts the following equations have.

(a) $s^4 + 8s^3 + 32s^2 + 80s + 100 = 0$.

(b) $s^5 + 10s^4 + 30s^3 + 80s^2 + 344s + 480 = 0$.

(c) $s^4 + 2s^3 + 7s^2 - 2s + 8 = 0$.

(d) $s^3 + s^2 + 20s + 78 = 0$.

(e) $s^4 + 6s^2 + 25 = 0$.

Solution:

(a)

$$s^4 + 8s^3 + 32s^2 + 80s + 100 = 0$$

$$\begin{array}{lclll} s^4 & : & 1 & 32 & 100 \\ s^3 & : & 8 & 80 & \\ s^2 & : & 22 & 100 & \\ s^1 & : & 80 - \dfrac{800}{22} = 43.6 & & \\ s^0 & : & 100 & & \end{array}$$

$\Longrightarrow$ No roots not in the LHP

(b)

$$s^5 + 10s^4 + 30s^3 + 80s^2 + 344s + 480 = 0$$

$$\begin{array}{lclll} s^5 & : & 1 & 30 & 344 \\ s^4 & : & 10 & 80 & 480 \\ s^3 & : & 22 & 296 & \\ s^2 & : & -545 & 480 & \\ s^1 & : & 490 & & \\ s^0 & : & 480 & & \end{array}$$

$\Longrightarrow$ 2 roots not in the LHP.

(c)

$$s^4 + 2s^3 + 7s^2 - 2s + 8 = 0$$

There are roots in the RHP (not all coefficients are >0).

$$\begin{array}{lcccc} s^4 & : & 1 & 7 & 8 \\ s^3 & : & 2 & -2 & \\ s^2 & : & 8 & 8 & \\ s^1 & : & -4 & & \\ s^0 & : & 8 & & \end{array}$$

$\Longrightarrow$ 2 roots not in the LHP.

(d) The Routh array is,

$$\begin{array}{lccc} s^3 & : & 1 & 20 \\ s^2 & : & 1 & 78 \\ s^1 & : & -58 & \\ s^0 & : & 78 & \end{array}$$

There are two sign changes in the first column of the Routh array. Therefore, there are two roots not in the LHP.

(e)

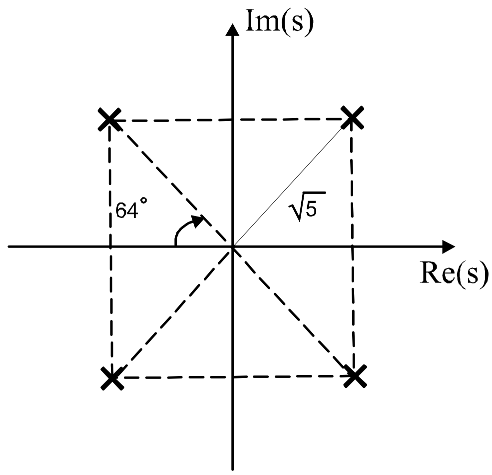
$$a(s) = s^4 + 6s^2 + 25 = 0$$

Two coefficients are missing so there are roots outside the LHP.
Create a new row by $\frac{da(s)}{ds}$

$$\begin{array}{lcccc} s^4 & : & 1 & 6 & 25 \\ s^3 & : & 4 & 12 & \longleftarrow \text{new row} \\ s^2 & : & 3 & 25 & \\ s^1 & : & 12 - \dfrac{100}{3} = -21.3 & & \\ s^0 & : & 25 & & \end{array}$$

$\Longrightarrow$ 2 roots not in the LHP
check:

$$\begin{aligned} a(s) &= 0 \Longrightarrow s^2 = -3 \pm 4j = 5e^{j(\pi \pm 0.92)} \\ s &= \sqrt{5}e^{j\left(\frac{\pi}{2} \pm 0.46\right) + n\pi j} \qquad n = 0, 1 \end{aligned}$$



Problem 3.40: s-plane pole locations

41. Find the range of K for which all the roots of the following polynomial are in the LHP.

$$s^5 + 5s^4 + 10s^3 + 10s^2 + 5s + K = 0.$$

Use MATLAB to verify your answer by plotting the roots of the polynomial in the s-plane for various values of K.

Solution:

$$s^5 + 5s^4 + 10s^3 + 10s^2 + 5s + K = 0$$

$$\begin{array}{lllll} s^5 & : & 1 & 10 & 5 \\ s^4 & : & 5 & 10 & K \\ s^3 & : & a_1 & a_2 & \\ s^2 & : & b_1 & K & \\ s^1 & : & c_1 & & \\ s^0 & : & K & & \end{array}$$

where

$$\begin{aligned} a_1 &= \frac{5\,(10) - 1\,(10)}{5} = 8 \qquad a_2 = \frac{5\,(5) - 1\,(K)}{5} = \frac{25 - K}{8} \\ b_1 &= \frac{(a_1)\,(10) - (5)\,(a_2)}{a_1} = \frac{55 + K}{8} \\ c_1 &= \frac{(b_1)\,(a_2) - (a_1)\,(K)}{b_1} = \frac{-\left(K^2 + 350K - 1375\right)}{5\,(55 + K)} \end{aligned}$$

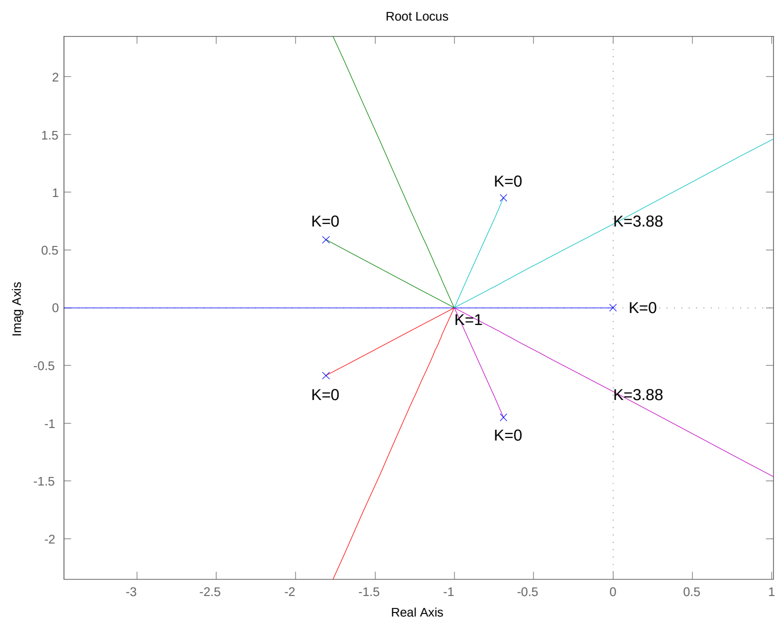
For stability: all terms in first column > 0

(1) $b_1 = \frac{55+K}{8} > 0 \Longrightarrow K > -55$

(2) $c_1 = \frac{-\left(K^2+350K-1375\right)}{5(55+K)} > 0, \frac{-(K-3.88)(K+354)}{5(55+K)} > 0 \Longrightarrow -55 < K < 3.88$

(3) $d_1 = K > 0$

Combining (1), (2), and (3) $\Longrightarrow 0 < K < 3.88$. If we plot the roots of the polynomial for various values of K we obtain the following root locus plot (see Chapter 5),



Problem 3.41: s-plane

42. The transfer function of a typical tape-drive system is given by

$$G(s) = \frac{K(s+4)}{s[(s+0.5)(s+1)(s^2+0.4s+4)]},$$

where time is measured in milliseconds. Using Routh's stability criterion, determine the range of K for which this system is stable when the characteristic equation is $1 + G(s) = 0$.

Solution:

$$1 + G(s) = s^5 + 1.9s^4 + 5.1s^3 + 6.2s^2 + (2+K)s + 4K = 0$$

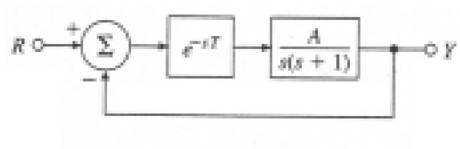


Figure 3.69: Control system for Problem 3.43

$$
\begin{array}{lllll}
s^5 & : & 1.0 & 5.1 & 2+K \\
s^4 & : & 1.9 & 6.2 & 4K \\
s^3 & : & a_1 & a_2 & \\
s^2 & : & b_1 & 4K & \\
s^1 & : & c_1 & & \\
s^0 & : & 4K & &
\end{array}
$$

where

$$
\begin{aligned}
a_1 &= \frac{(1.9)(5.1)-(1)(6.2)}{1.9} = 1.837 \qquad a_2 = \frac{(1.9)(2+K)-(1)(4K)}{1.9} = 2-1.1K \\
b_1 &= \frac{(a_1)(6.2)-(a_2)(1.9)}{a_1} = 1.138(K+3.63) \\
c_1 &= \frac{(b_1)(a_2)-(4K)(a_1)}{b_1} = \frac{-\left(1.25K^2+9.61K-8.26\right)}{1.138(K+363)} = \frac{-(K+8.47)(K-0.78)}{0.91(K+3.63)}
\end{aligned}
$$

For stability:

$$
\begin{aligned}
(1)\ b_1 &= K+3.63>0 \Longrightarrow K>-3.63 \\
(2)\ c_1 &> 0 \Longrightarrow -8.43<K<0.78 \\
(3)\ d_1 &> 0 \Longrightarrow K>0
\end{aligned}
$$

Intersection of (1), (2), and (3) $\Longrightarrow 0<K<0.78$

43. Consider the system shown in Fig. 3.69.

    (a) Compute the closed-loop characteristic equation.

    (b) For what values of (T, A) is the system stable? Hint: An approximate answer may be found using

$$e^{-Ts} \cong 1-Ts$$

or

$$e^{-Ts} \cong \frac{1 - \frac{T}{2}s}{1 + \frac{T}{2}s}$$

for the pure delay. As an alternative, you could use the computer MATLAB (Simulink) to simulate the system or to find the roots of the system's characteristic equation for various values of T and A.

Solution:

(a) The characteristic equation is,

$$s(s+1) + Ae^{-Ts} = 0$$

(b) Using $e^{-Ts} \cong 1 - Ts$, the characteristic equation is,

$$s^2 + (1 - TA)s + A = 0$$

The Routh's array is,

$$\begin{array}{lcll} s^2 & : & 1 & A \\ s^1 & : & 1 - TA & 0 \\ s^0 & : & A & \end{array}$$

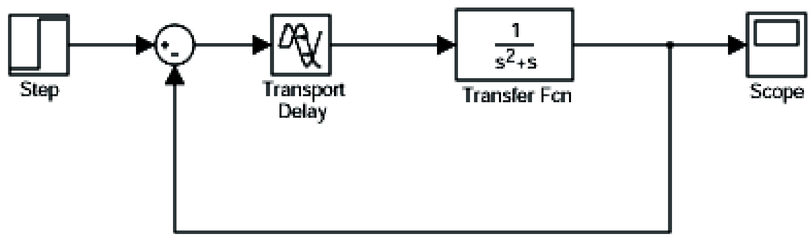
For stability we must have $A > 0$ and $TA < 1$.
Using $e^{-Ts} \cong \frac{(1-\frac{T}{2}s)}{(1+\frac{T}{2}s)}$, the characteristic equation is,

$$s^3 + (1 + \frac{2}{T})s^2 + (\frac{2}{T} - A)s + \frac{2}{T}A = 0$$

The Routh's array is,

$$\begin{array}{lcll} s^3 & : & 1 & (\frac{2}{T} - A) \\ s^2 & : & (1 + \frac{2}{T}) & \frac{2A}{T} \\ s^1 & : & \frac{(1+\frac{2}{T})(\frac{2}{T} - A) - \frac{2A}{T}}{(1+\frac{2}{T})} & 0 \\ s^0 & : & \frac{2A}{T} & \end{array}$$

For stability we must have all the coefficients in the first column be positive.
The following Simulink diagram simulates the system.



Problem 3.43: Simulink simulation diagram

44. Modify the Routh criterion so that it applies to the case where all the poles are to be to the left of $-\alpha$ when $\alpha > 0$. Apply the modified test to the polynomial

$$s^3 + (6+K)s^2 + (5+6K)s + 5K = 0,$$

finding those values of K for which all poles have a real part less than -1.

Solution:

Let $p = s + \alpha$ and substitute $s = p - \alpha$ to obtain a polynomial in terms of p. Apply the standard Routh test to the polynomial in p.

For the example $p = s+1$ or $s = p-1$. Substitute this in the polynomial,

$$(p-1)^3 + (6+K)(p-1)^2 + (5+6K)(p-1) + 5K = 0$$

or

$$p^3 + (3+K)p^2 + (4K-4)p + 1 = 0.$$

The Routh's array is,

$$\begin{array}{llll} p^3 & : & 1 & 4K-4 \\ p^2 & : & 3+K & 1 \\ p^1 & : & \dfrac{(3+K)(4K-4)-1}{3+K} & 0 \\ p^0 & : & 1 & \end{array}$$

We must have $K > -3$ and $4K^2 + 8K - 13 > 0$. The roots of the second-order polynomial are $K = 1.06$ and $K = -3.061$. The second-order polynomial remains positive if $K > 1.06$ or $K < -3.061$. Therefore, we must have $K > 1.06$.

45. Suppose the characteristic polynomial of a given closed-loop system is computed to be

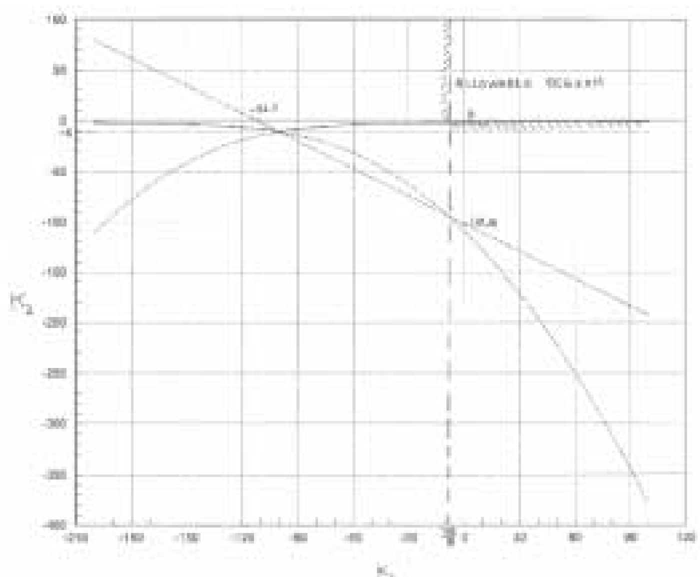
$$s^4+(11+K_2)s^3+(121+K_1)s^2+(K_1+K_1K_2+110K_2+210)s+11K_1+100=0.$$

Find constraints on the two gains K_1 and K_2 that guarantee a stable closed-loop system, and plot the allowable region(s) in the (K_1, K_2) plane. You may wish to use the computer to help solve this problem.

Solution:

$$\begin{array}{lllll}
s^5 & : & & & \\
s^4 & : & 1 & 121+K_1 & 11K_1+100 \\
s^3 & : & 11+K_2 & K_1+K_1K_2+110K_2+210 & 0 \\
s^2 & : & \dfrac{(11K_2+10K_1+1121)}{K_2+11} & 11K_1+100 & \\
s^1 & : & \multicolumn{3}{l}{\dfrac{10\left(111K_2^2+K_1^2K_2+199K_1K_2+12342K_2+K_1^2+189K_1+22331\right)}{(11K_2+10K_1+1121)}} \\
s^0 & : & 11K_1+100 & &
\end{array}$$

For stability the first column must be all positive. This means that $K_2 > -11$ and $K_1 > -\frac{100}{11}$. The region of stability is shown in the following figure.



Problem 3.45: s-plane region

46. Overhead electric power lines sometimes experience a low-frequency, high-amplitude vertical oscillation, or gallop, during winter storms when the line conductors become covered with ice. In the presence of wind, this ice can assume aerodynamic lift and drag forces that result in a gallop up to several meters in amplitude. Large-amplitude gallop can cause clashing conductors and structural damage to the line support structures caused by the large dynamic loads. These effects in turn can lead to power outages. Assume that the line conductor is a rigid rod, constrained to vertical motion only, and suspended by springs and dampers as shown in Fig. 3.70. A simple model of this conductor galloping is

$$m\ddot{y} + \frac{D(\alpha)\dot{y} - L(\alpha)v}{(\dot{y}^2 + v^2)^{1/2}} + T\left(\frac{n\pi}{\ell}\right)y = 0,$$

where

$$\begin{aligned}
m &= \text{mass of conductor},\\
y &= \text{conductor's vertical displacement},\\
D &= \text{aerodynamic drag force},\\
L &= \text{aerodynamic lift force},\\
v &= \text{wind velocity},\\
\alpha &= \text{aerodynamic angle of attack} = -\tan^{-1}(\dot{y}/v),\\
T &= \text{conductor tension},\\
n &= \text{number of harmonic frequencies},\\
\ell &= \text{length of conductor}.
\end{aligned}$$

Assume that $L(0) = 0$ and $D(0) = D_0$ (a constant), and linearize the equation around the value $y = \dot{y} = 0$. Use Routh's stability criterion to show that galloping can occur whenever

$$\frac{\partial L}{\partial \alpha} + D_0 < 0.$$

Solution:

$$m\ddot{y} + \left[\frac{D(\alpha)\,\dot{y} - L(\alpha)\,v}{\sqrt{\dot{y}^2 + v^2}}\right] + T\left(\frac{n\pi}{l}\right)^2 y = 0,$$

Let $x_1 = y$ and $x_2 = \dot{y} = \dot{x}_1$

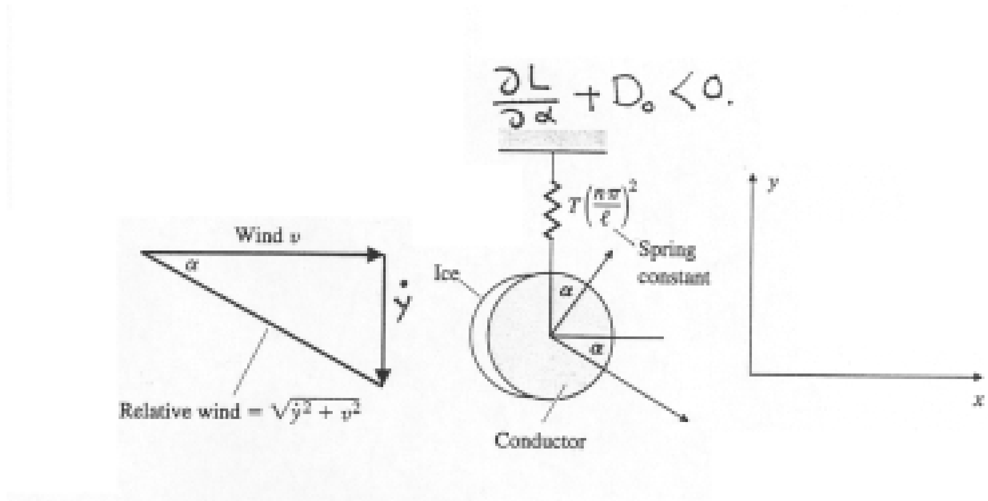


Figure 3.70: Electric power-line conductor

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{1}{m}\left[\frac{D(\alpha)\,x_2 - L(\alpha)\,v}{\sqrt{x_2^2+v^2}}\right] - \frac{T}{m}\left(\frac{n\pi}{l}\right)^2 x_1 = 0 \\
\alpha &= -\tan^{-1}\left(\frac{x_2}{v}\right) \\
\dot{x}_1 &= f_1(x_1, x_2) \\
\dot{x}_2 &= f_2(x_1, x_2)
\end{aligned}
$$

$$
\begin{aligned}
\dot{x}_1 &= \dot{x}_2 = 0 \text{ implies } x_2 = 0 \\
x_2 &= 0 \text{ implies } \alpha = 0 \\
\alpha &= 0 \text{ implies } -\frac{T}{m}\left(\frac{n\pi}{l}\right)^2 x_1 = 0 \text{ implies } x_1 = 0.
\end{aligned}
$$

$$
\frac{\partial f_1}{\partial x_1} = 0, \qquad \frac{\partial f_2}{\partial x_2} = 1, \qquad \frac{\partial f_2}{\partial x_1} = -\frac{T}{m}\left(\frac{n\pi}{l}\right)^2
$$

$$\frac{\partial f_2}{\partial x} = \frac{\partial}{\partial x_2}\left\{-\frac{1}{m}\left[\frac{D(\alpha)\,x_2 - L(\alpha)\,v}{\sqrt{x_2^2+v^2}}\right]\right\}$$
$$= -\frac{1}{m}\{\frac{1}{\sqrt{x_2^2+v^2}}\left[\frac{\partial D}{\partial\alpha}\frac{\partial\alpha}{\partial x_2}x_2 + D(\alpha) - \frac{\partial L}{\partial\alpha}\frac{\partial\alpha}{\partial x_2}\right]$$
$$-\left[\frac{D(\alpha)\,x_2 - L(\alpha)\,v}{\sqrt{x_2^2+v^2}}\right]\left[\frac{-x_2}{(x_2^2+v^2)^{\frac{3}{2}}}\right]\}$$

Now

$$\frac{\partial\alpha}{x_2} = \frac{\partial}{\partial x_2}\left(-\tan^{-1}\left(\frac{x_2}{v}\right)\right) = \frac{-1}{1+\frac{x_2^2}{v^2}}\left(\frac{1}{v}\right)$$

so

$$\frac{\partial f_2}{\partial x_2} = \frac{-1}{m}\{\frac{1}{\sqrt{x_2^2+v^2}}\left[\frac{-\frac{\partial D}{\alpha}x_2}{v\left(1+\frac{x_2^2}{v^2}\right)} + D(\alpha) + \frac{\frac{\partial L}{\partial\alpha}v}{v\left(1+\frac{x_2^2}{v^2}\right)}\right]$$
$$-\left[\frac{D(\alpha)\,x_2 - L(\alpha)\,v}{\sqrt{x_2^2+v^2}}\right]\left[\frac{-x_2}{(x_2^2+v^2)^{\frac{3}{2}}}\right]\}$$
$$\frac{\partial f_2}{\partial x_2}|_{x_2=0} = -\frac{1}{m}\left\{\frac{1}{v}\left[D_0 + \frac{\partial L}{\partial\alpha}\right]\right\} = -\frac{1}{mv}\left(D_0 + \frac{\partial L}{\partial\alpha}\right)$$

For no damping (or negative damping) δx_2 term must be ≤ 0 so this implies $D_0 + \frac{\partial L}{\partial\alpha} < 0$.

Problems and Solutions for Section 3.7

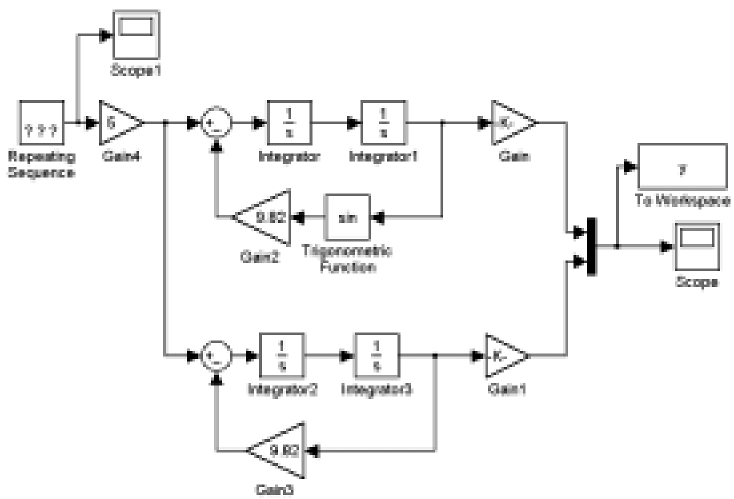
47. Repeat Example 3.34 using Simulink.

Solution:

The following Simulink diagram generates the same transient response as in the MATLAB implementation.

Table 3.1: Step-Response Data for Problem 3.48

t	$y(t)$	t	$y(t)$	t	$y(t)$
0	0	0.20	0.0138	0.90	0.4409
0.02	0.0001	0.22	0.0395	1.00	0.4924
0.04	0.0005	0.24	0.0480	1.50	0.6904
0.06	0.0014	0.26	0.0571	2.00	0.8121
0.08	0.0031	0.28	0.0668	2.50	0.8860
0.10	0.0057	0.30	0.0771	3.00	0.9309
0.12	0.0091	0.50	0.1979	3.50	0.9581
0.14	0.0135	0.60	0.2624	4.00	0.9746
0.16	0.0187	0.70	0.3253	5.00	0.9907
0.18	0.0248	0.80	0.3851		



Problem 3.47: Simulink diagram

Problems and Solutions for Section 3.8

48. Samples from a step response are given in Table 3.1. Plot this data on a linear scale [$y(t)$ vs. t] and semilog scale [$\log(y - y_\infty)$ vs. t], and obtain an estimate of the transfer function.

Solution:

Model from a step response data:

$$y = y_\infty + Ae^{-\alpha t} + Be^{-\beta t} + \ldots$$

First approximation: $y - y_\infty = Ae^{-\alpha t}$

$$\log\left(|y - y_\infty|\right) = \log_{10}|A| - 0.4343\alpha t$$

From the line fit to the following plot find A and α.

$$\begin{aligned} A &= -1.35 \\ 0.4343\alpha &= \frac{0.733 - 0.3}{1} = 0.433 \\ \alpha &\backsimeq 1 \end{aligned}$$

Second term: $y - 1 + 1.35e^{-t} = Be^{-\beta t}$

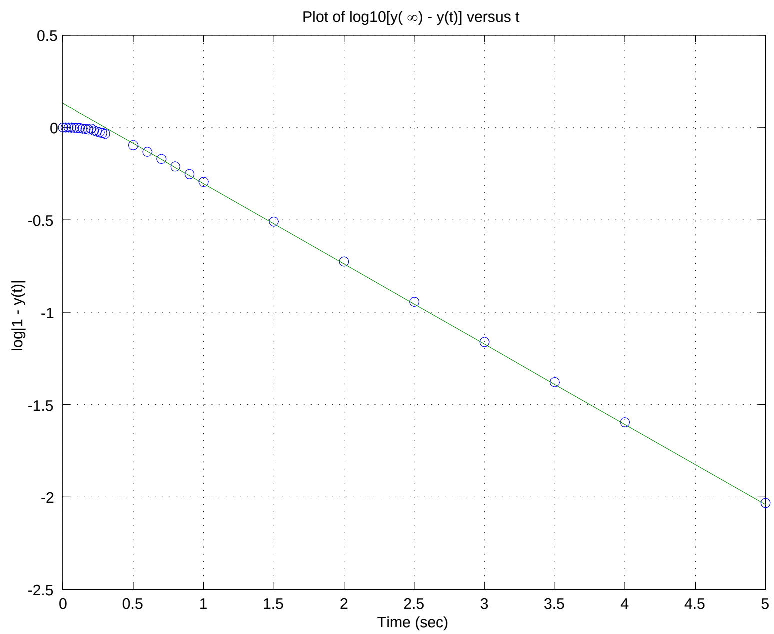
From the line $B = 0.35$ $\beta = 5$

The model: $\hat{y} = 1 - 1.35e^{-t} + 0.35e^{-5t}$ comparing $\hat{y}$ to y shows good approximation for $t > 0.3$ (better than 5%).

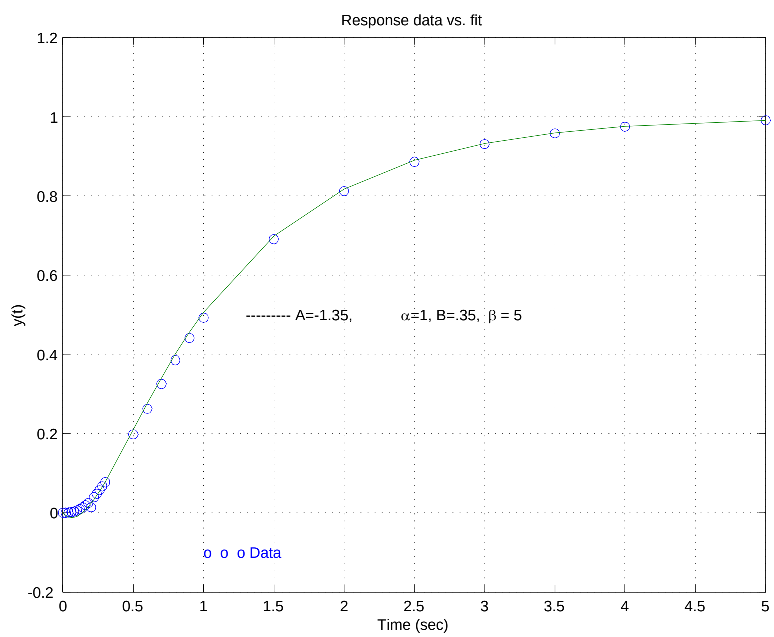
The transfer function for the above model:

$$Y(s) = \frac{1}{s} - \frac{1.35}{s+1} + \frac{0.35}{s+5} = \frac{s - 0.45}{s\,(s+1)\,(s+5)}$$

t	y	$\hat{y}$
0	0	0
0.2	0.0138	0.0235
0.3	0.0771	0.078
0.5	0.1979	0.21
1	0.4924	0.506
2	0.8121	0.817
3	0.9309	0.933
5	0.9907	0.9909



Problem 3.49: Plot of log10[y(∞)-y(t)] versus t



Problem 3.49: Response data vs. fit